MS221 Chapter C1

The Open University

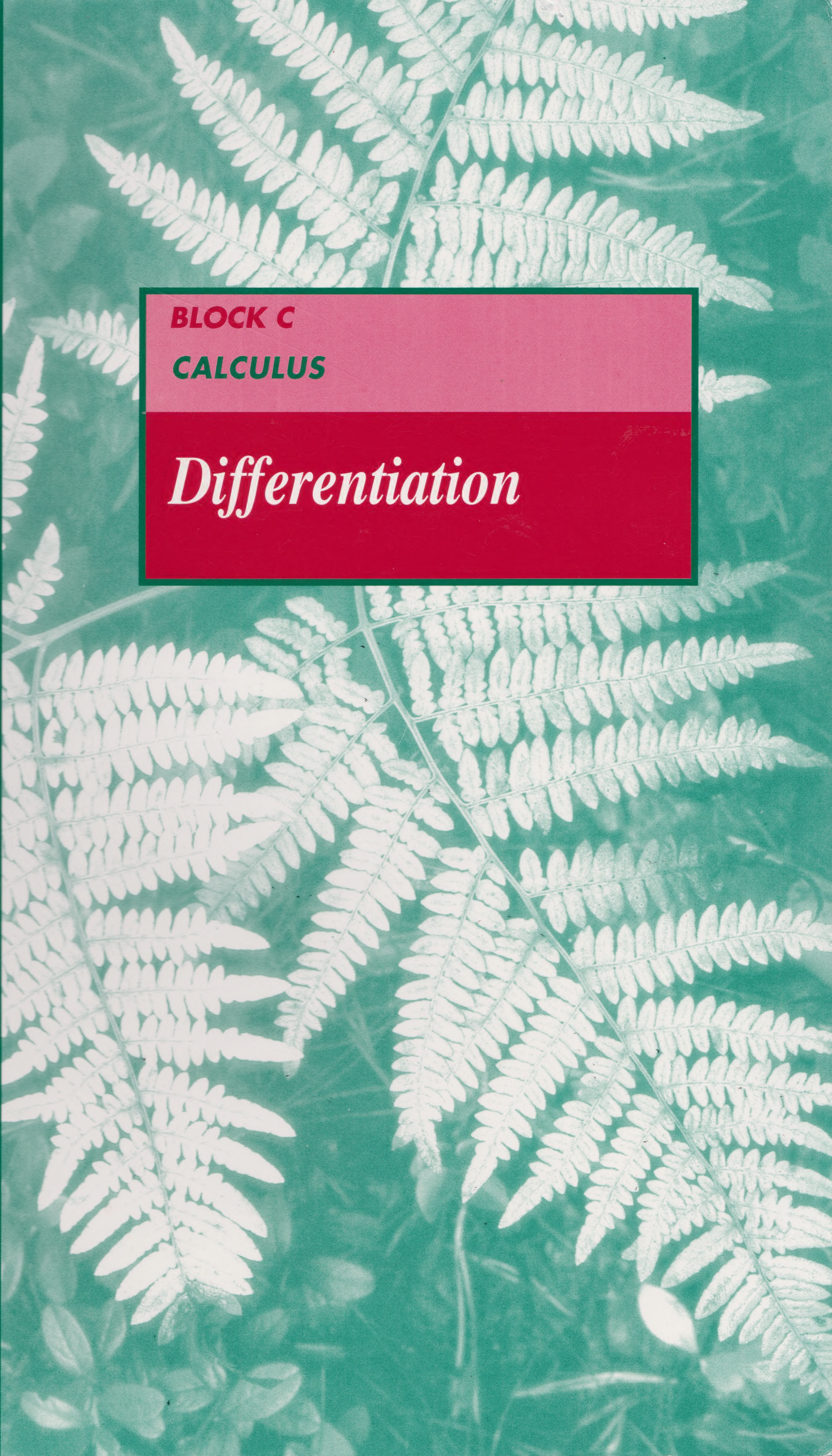
A second level interdisciplinary course

Exploring Mathematics

BLOCK C

CALCULUS

Differentiation



CHAPTER

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The Open University

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Exploring **Mathematics**

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CALCULUS

Differentiation

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Prepared by the course team

About this course

This course, MS221 *Exploring Mathematics*, and the courses MU120 *Open Mathematics* and MST121 *Using Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MS221 uses the software program Mathcad (MathSoft, Inc.) to investigate mathematical concepts and as a tool in problem solving. This software is provided as part of the course.

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Contents

Introduction to Block C

Block C is concerned with the subject of *calculus*. In mathematics, the word 'calculus' means 'a systematic method for solving a certain type of problem'. But the word is most commonly used as a shorthand for the two related topics of *differential calculus* and *integral calculus*. These topics were developed in the seventeenth century, and since then they have been key tools in many branches of mathematics, science and technology.

Here are some examples of the types of problem that calculus can help us to solve.

1. How can we find the shape of the graph of a function and, in particular, how can we locate the points where the function takes its maximum and minimum values?
2. How can we find the area of the plane that lies under the graph of a function and above the x-axis, between two particular values on the x-axis?
3. How can we approximate non-polynomial functions such as cos, ln and arctan to suitable accuracy with polynomial functions?

It turns out that these three apparently different problems have methods of solution that are closely related to each other, and these methods form the core topics of calculus.

The first problem, about the shape of the graph of a function, is solved by using the process called *differentiation*, which enables us to find a formula for the gradient of the graph of the function. This topic is covered in Chapter C1.

You met a formula for the gradient of a general quadratic function in Chapter B1.

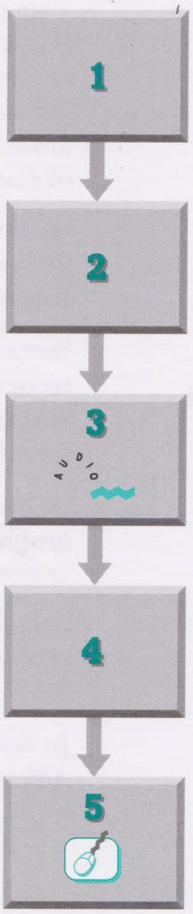
The second problem, about the area under the graph of a function, is solved by using a process called *integration*, which is the opposite of differentiation, in a certain sense. This topic is covered in Chapter C2.

The third problem about approximating non-polynomial functions by polynomial functions is solved in Chapter C3 by using both differentiation and integration.

The history of the development of calculus in the seventeenth century is of considerable interest. The major figures involved were Sir Isaac Newton (1642–1727), in England, and Gottfried Wilhelm Leibniz (1646–1716), in Germany. But many other mathematicians played a role in the development, including Pierre de Fermat (1601–1665), in France.

There is an *optional* video band on DVD00115, band C(i) 'The birth of calculus', associated with this historical development, which you can watch at any time during your study of Block C.

Study guide



There are five sections to this chapter. They are intended to be studied consecutively in five study sessions.

Section 3 requires the use of an audio CD player, and Section 5 requires the use of the computer together with Computer Book C.

The pattern of study for each session might be as follows.

Study session 1: Section 1.
Study session 2: Section 2.
Study session 3: Section 3.
Study session 4: Section 4.
Study session 5: Section 5.

Each session will require two to three hours, the longer ones being the second and third.

The material in this chapter is self-contained. However, it is likely that you already have some familiarity with the following topics:

- ◇ definition of the derivative of a function f, and its notations $f'(x)$ and $\dfrac{dy}{dx}$ where $y = f(x)$;
- ◇ differentiating basic functions;
- ◇ rules for differentiating sums, constant multiples, products, quotients and composites of functions;
- ◇ stationary points, and the identification and location of local maxima and local minima of a function.

These topics are in MST121 Chapter C1.

The optional Video Band C(ii) 'Algebra Workout – Differentiation' could be viewed at any stage during your study of this chapter.

Introduction

This chapter is concerned with the process of finding the gradient of a graph of a function at a given point on the graph. The value of this gradient is called the *derivative* of the function at that point. By finding a formula for the derivative of a function and working with this, we can make useful deductions about the behaviour of the function. The process of finding the derivative of a function is called *differentiation*.

Section 1 defines differentiation, introducing various notations for the derivative, and obtains the derivatives of several important basic functions. Higher derivatives, arising from repeated differentiation, are also defined.

Section 2 describes several rules for differentiating functions that can be recognised as sums, constant multiples, products, quotients, composites and inverses of other functions. These rules greatly extend the range of functions that can be differentiated.

In Section 3 the process of differentiation is applied to help sketch the graphs of certain types of function, and a general graph-sketching strategy is given.

Section 4 describes an important application of the derivative to the problem of finding approximate solutions to equations of the form $f(x) = 0$; this application is the so-called *Newton–Raphson method.*

In Section 5 you will see that the computer can be used to carry out differentiation, and also to implement the Newton–Raphson method.

1 Differentiation

Earlier in the course you met a formula for the gradient of the graph of a quadratic function at a point on the graph; that is, the gradient of the tangent to the graph at the point. It was shown that for the general quadratic function $f(x) = ax^2 + bx + c$, the gradient of the graph of $y = f(x)$ at the point $(x, f(x))$ on the graph is given by the formula

See Chapter B1, Subsection 2.1.

The tangent is the unique line that 'touches' the graph at the point.

$$f'(x) = 2ax + b. \tag{1.1}$$

For example, if $f(x) = x^2$, then $f'(x) = 2x$, so the gradient of $y = x^2$ at the point $(1, 1)$, say, is $f'(1) = 2$, as illustrated in Figure 1.1.

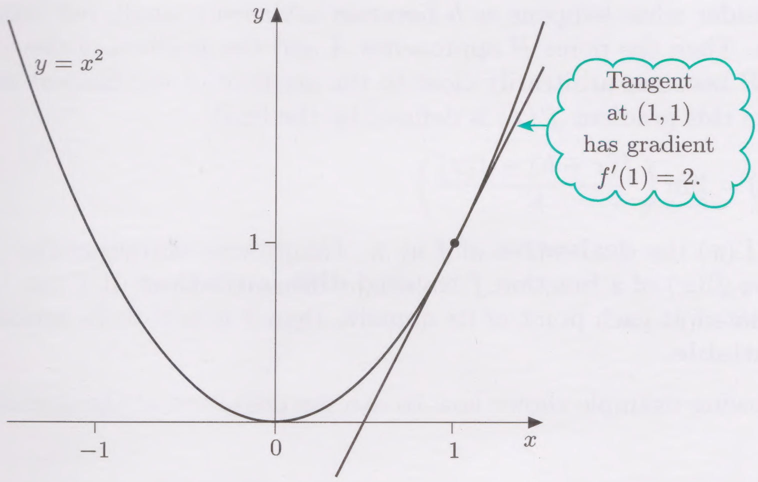


Figure 1.1 Graph of $f(x) = x^2$ and its tangent at $(1, 1)$

In this section we find such 'gradient formulas' for a wide range of basic functions.

1.1 Derivatives

The notation $f'(x)$ is used generally to denote the gradient of the graph of a function f at the point $(x, f(x))$, whenever this gradient exists. For a given function f, we define $f'(x)$ as follows. Let $A(x, f(x))$ and $B(x + h, f(x + h))$ be two points on the graph of $y = f(x)$. The chord that joins these two points has gradient given by

$$\frac{f(x+h) - f(x)}{h}.$$

If A and B are nearby points, so that h is small, then the chord AB approximates the tangent at A, as shown in Figure 1.2 overleaf.

In Figure 1.2 we have $h > 0$, so $x + h$ lies to the right of x. If $h < 0$, then $x + h$ lies to the left of x, but the gradient of the chord is given by the same formula.

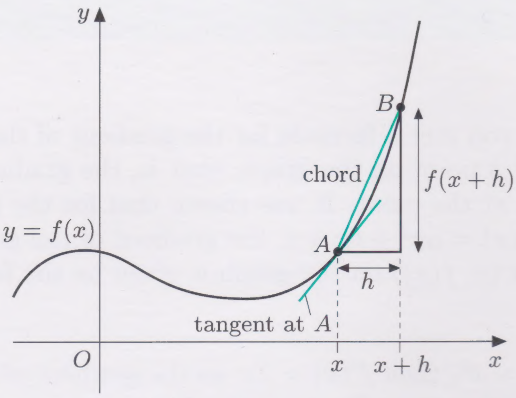


Figure 1.2 A chord that approximates the tangent at A

Now consider what happens as h becomes arbitrarily small, but remains non-zero. Then the point B approaches A and the gradient of the chord AB becomes arbitrarily close to the gradient of the tangent at A. Therefore this gradient $f'(x)$ is defined by the limit

We read the expression

$$\lim_{h \to 0}$$

as 'the limit as h tends to 0'.

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right). \tag{1.2}$$

We call $f'(x)$ the **derivative** of f at x. The process of finding the derivative $f'(x)$ of a function f is called **differentiation**. If f can be differentiated at each point of its domain, then f is said to be **smooth** or **differentiable**.

In this chapter, 'smooth' is usually used.

The following example shows how to use the definition of the derivative.

Example 1.1 Differentiation from the definition

(a) Use equation (1.2) to differentiate the function $f(x) = x^4$.

(b) What is the gradient of the graph of $y = x^4$ at the point $(-2, 16)$?

Solution

(a) To find $f'(x)$, the derivative of f at x, we consider the quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^4 - x^4}{h},$$

where h is a non-zero number. We need to find the limit of this quotient as h tends to 0. By the Binomial Theorem,

See Chapter B1, Subsection 5.3.

$$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4,$$

so

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4) - x^4}{h} \\ &= 4x^3 + 6x^2h + 4xh^2 + h^3. \end{aligned}$$

Now, as h tends to 0 each of the terms $6x^2h$, $4xh^2$ and h^3 tends to 0, so

$$\lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3. \tag{1.3}$$

Thus, by equation (1.2), the derivative of f at x is $f'(x) = 4x^3$.

(b) For the point $(-2, 16)$ we have $x = -2$. Thus, by part (a), the gradient at $(-2, 16)$ is

$$f'(-2) = 4 \times (-2)^3 = -32.$$

Comment

In deducing equation (1.3), two unstated 'rules' for limits were used. These rules concern limits of functions which are 'sums' and 'constant multiples', and they can be stated informally as follows:

$$\lim\,(F+G) = \lim F + \lim G$$

and

$$\lim(cF) = c\lim F.$$

Here F and G are functions and c is a constant. We often use such 'obvious' rules when working with limits, but we shall not refer to them explicitly, nor to the rules for limits of functions which are 'products' or 'quotients':

$$\lim(FG) = (\lim F)(\lim G)$$

and

$$\lim\left(\frac{F}{G}\right) = \frac{\lim F}{\lim G}, \quad \text{provided that } \lim G \neq 0.$$

In the following activity you are asked to use equation (1.2) to differentiate the function $f(x) = 1/x^2$, whose graph is shown in Figure 1.3.

Activity 1.1 Differentiation from the definition

Consider the function $f(x) = 1/x^2$ and suppose that $x \neq 0$.

(a) Show that if $h \neq 0$ and $x + h \neq 0$, then

$$\frac{f(x+h) - f(x)}{h} = \frac{-2x-h}{(x+h)^2x^2}.$$

(b) Hence determine $f'(x)$.

(c) Calculate $f'(2)$ and $f'(-1)$, and sketch the corresponding tangents in Figure 1.3.

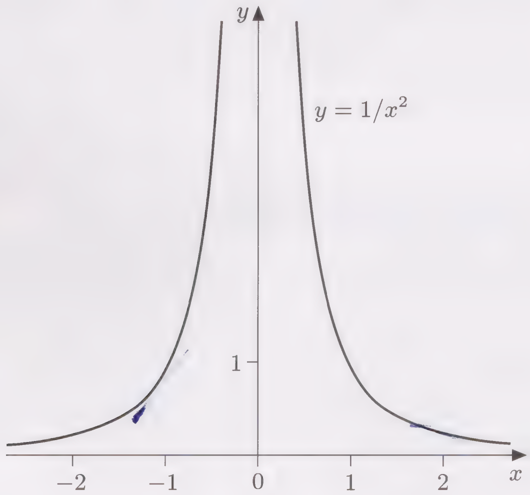


Figure 1.3 Graph of $f(x) = 1/x^2$

Solutions are given on page 52.

The derivative of a function f defines a new function, denoted by f'; this function is called the **derived function** of f. For example, from Activity 1.1 it follows that the derived function of $f(x) = 1/x^2$ is

$$f'(x) = -\frac{2}{x^3}.$$

In practice, the domain of the derived function f' of a function f consists of those points in the domain of f where the derivative of f exists.

The domain of the function $f(x) = 1/x^2$ is $\mathbb{R}$ except 0, by the domain convention (see Chapter B1, Subsection 1.1). This set is also the domain of $f'(x) = -2/x^3$.

Leibniz notation

There is another notation in common use for derivatives, due to Leibniz. If y is a dependent variable which is expressed as 'a function of' the independent variable x in the form $y = f(x)$, then the derivative $f'(x)$ is written as $\dfrac{dy}{dx}$. For example,

$$\text{if } y = x^2, \text{ then } \frac{dy}{dx} = 2x.$$

Often the word 'derivative' is also used to refer to the 'derived function'.

The symbol $\dfrac{dy}{dx}$ is a complete symbol, not a fraction, and is pronounced as 'dee-y by dee-x'.

Informally, $\dfrac{dy}{dx}$ is called 'the derivative of y with respect to x'. This notation arises from the definition of the derivative given by Leibniz, which is

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}.$$

Here δx denotes a small change in the variable x and δy denotes the corresponding change in the variable y. Thus the derivative can be interpreted as the instantaneous rate of change of y with respect to x.

This matches our previous definition of $f'(x)$ in equation (1.2), with h replaced by δx and $f(x+h) - f(x)$ replaced by δy. Note that δx and δy are complete symbols, not products.

A useful variation of the Leibniz notation is

$$\frac{d}{dx}(f(x)) \quad \text{to mean} \quad f'(x).$$

When the derivative for a particular value of x is required, these notations can be adapted as follows:

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{d}{dx}(f(x))\right|_{x=a}.$$

These notations apply also to other variables. For example, if $s = f(t)$, then

$$\frac{ds}{dt} = \frac{d}{dt}(f(t)) = f'(t).$$

When t represents time, the derivative $f'(t)$ is sometimes written as $\dot{s}$, which is an example of Newton's notation.

1.2 Some basic derivatives

In this subsection, the derivatives of various basic functions are obtained.

Constant functions

The graph of a constant function is horizontal, so the tangent at each point is horizontal; see Figure 1.4.

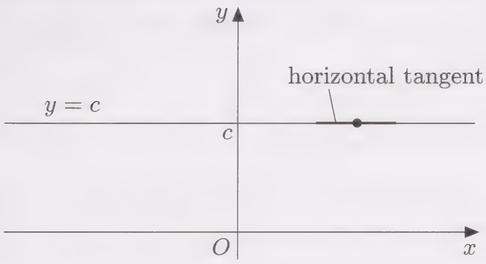


Figure 1.4 Graph of $f(x) = c$

Thus the gradient of this graph is 0 at all points.

> If $f(x) = c$, where c is a constant, then $f'(x) = 0$.

Power functions

You have seen several examples of the derivatives of power functions, which have the form $f(x) = x^n$, where $n \neq 0$; for example,

$$\text{if } f(x) = x^4, \text{ then } f'(x) = 4x^3.$$

The general rule for differentiating a power function is as follows.

> If $f(x) = x^n$, where $n \neq 0$, then $f'(x) = nx^{n-1}$.

Here the domain of f depends on n; for example, if $n = 4$, then the domain is $\mathbb{R}$, whereas if $n = -\frac{1}{2}$, then the domain is $(0, \infty)$.

For example, for the power function

$$f(x) = \frac{1}{\sqrt{x}} = x^{-1/2},$$

the power is $n = -\frac{1}{2}$, so

$$f'(x) = -\tfrac{1}{2}x^{(-1/2)-1} = -\tfrac{1}{2}x^{-3/2} = -\frac{1}{2x^{3/2}}.$$

This example illustrates the fact that, although we usually avoid negative powers when displaying the rule of a function, such powers are needed when finding derivatives.

Activity 1.2 Differentiating powers

Differentiate each of the following functions.

(a) $f(x) = x^{100}$ (b) $f(x) = \dfrac{1}{x^{3/2}}$

Solutions are given on page 52.

In the case when n is a positive integer, the rule for differentiating powers can be justified by using the Binomial Theorem. If $f(x) = x^n$ and h is a non-zero number, then

Other cases are justified later in the chapter.

If $n = 1$, the calculation is simple:
$$\frac{(x+h)-x}{h} = 1.$$

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^n - x^n}{h}\\ &= \frac{(x^n + {}^nC_1x^{n-1}h + {}^nC_2x^{n-2}h^2 + \cdots + h^n) - x^n}{h}\\ &= {}^nC_1x^{n-1} + {}^nC_2x^{n-2}h + \cdots + h^{n-1}\\ &= nx^{n-1} + {}^nC_2x^{n-2}h + \cdots + h^{n-1}, \qquad (1.4)\end{aligned}$$

since ${}^nC_1 = n$. In the final expression in equation (1.4), all terms except the first include a positive power of h and so they each tend to zero as h tends to zero. Thus

As $h \to 0$, x remains constant.

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = nx^{n-1},$$

as required.

Sine and cosine functions

Next, consider the sine and cosine functions. The limits and algebra involved here are not straightforward, but try to follow the reasoning given to justify the result for the sine function.

You will not be expected to produce such reasoning.

> If $f(x) = \sin x$, then $f'(x) = \cos x$.

We have to consider the quotient
$$\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h) - \sin x}{h},$$
where h is non-zero. Since $\sin(x+h) = \sin x \cos h + \cos x \sin h$, we obtain

See Chapter A3, Subsection 3.2.

$$\frac{f(x+h)-f(x)}{h} = \frac{(\sin x\cos h + \cos x \sin h) - \sin x}{h}.$$

There is no convenient cancellation of h in the right-hand expression, as in earlier cases. Instead we collect like terms to give
$$\frac{f(x+h)-f(x)}{h} = \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right). \qquad (1.5)$$

Now it will shortly be shown that
$$\lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) = 0 \quad \text{and} \quad \lim_{h\to 0}\left(\frac{\sin h}{h}\right) = 1. \qquad (1.6)$$

You may like to evaluate these quotients for $h = 0.01$ and $h = 0.001$, say, to gain an idea of their behaviour for small values of h.

Using these limits we obtain, from equation (1.5),
$$\begin{aligned}f'(x) &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)\\ &= \sin x \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \cos x \lim_{h\to 0}\left(\frac{\sin h}{h}\right)\\ &= \sin x \times 0 + \cos x \times 1\\ &= \cos x,\end{aligned}$$

as required.

To evaluate the two limits in equation (1.6), we use a geometric argument. Consider a circle of radius 1 and a small acute angle of θ radians at the centre, subtended by an arc AB, as shown in Figure 1.5(a).

We use θ rather than h since this is a geometric argument.

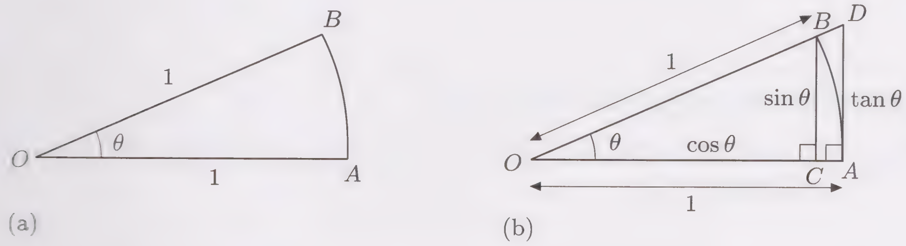


Figure 1.5 (a) Sector OBA (b) Triangles OBC and ODA

Now draw lines BC and DA perpendicular to OA, with OBD forming a straight line, as shown in Figure 1.5(b). Then triangles OBC and ODA are right-angled. In ΔOBC, OB has unit length so BC has length $\sin\theta$ and OC has length $\cos\theta$. In ΔODA, OA has unit length so AD has length $\tan\theta$. From Figure 1.5(b), we can see that

area of ΔOBC < area of sector OBA < area of ΔODA;

that is,

$$\tfrac{1}{2}\sin\theta\cos\theta < \tfrac{1}{2}\theta < \tfrac{1}{2}\tan\theta.$$

The area of a sector of a circle of radius r subtending an angle θ at the centre is $\frac{1}{2}r^2\theta$.

On multiplying these inequalities by $2/\sin\theta$, we obtain

$$\cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Since $0 < \theta < \frac{1}{2}\pi$, we have $\sin\theta > 0$.

As $\theta \to 0$, we have $\cos\theta \to \cos 0 = 1$, so the quotient $\theta/\sin\theta$ is trapped between two expressions both of which tend to 1. Thus

$$\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right) = 1.$$

Hence

$$\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1, \qquad (1.7)$$

as in equation (1.6).

In this argument θ tends to 0 through *positive* values, but

$$\frac{\sin(-\theta)}{(-\theta)} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta},$$

so the result holds in general.

Next we evaluate $\lim\limits_{h\to 0}((\cos h - 1)/h)$. Here we use an algebraic argument, together with the limit in equation (1.7). First, recall the half-angle formula

$$\sin^2(\tfrac{1}{2}\theta) = \tfrac{1}{2}(1-\cos\theta).$$

See Chapter A3, Subsection 3.3.

This formula gives

$$\frac{1-\cos\theta}{\theta} = \frac{2\sin^2(\frac{1}{2}\theta)}{\theta} = \left(\frac{\sin(\frac{1}{2}\theta)}{\frac{1}{2}\theta}\right)^2 \times (\tfrac{1}{2}\theta).$$

Now, by equation (1.7),

$$\lim_{\theta\to 0}\left(\frac{\sin(\frac{1}{2}\theta)}{\frac{1}{2}\theta}\right) = 1, \quad \text{so} \quad \lim_{\theta\to 0}\left(\frac{1-\cos\theta}{\theta}\right) = 1^2 \times 0 = 0,$$

$\frac{1}{2}\theta \to 0$ as $\theta \to 0$, and vice versa.

as stated in equation (1.6).

Similar reasoning can be used with the cosine function to give the following result; the details are omitted.

> If $f(x) = \cos x$, then $f'(x) = -\sin x$.

Exponential and logarithm functions

Figure 1.6 shows the graphs of the functions $f(x) = a^x$, for the values $a = 1$, $a = 2$ and $a = 3$.

See MST121 Chapter A3, Subsection 3.2.

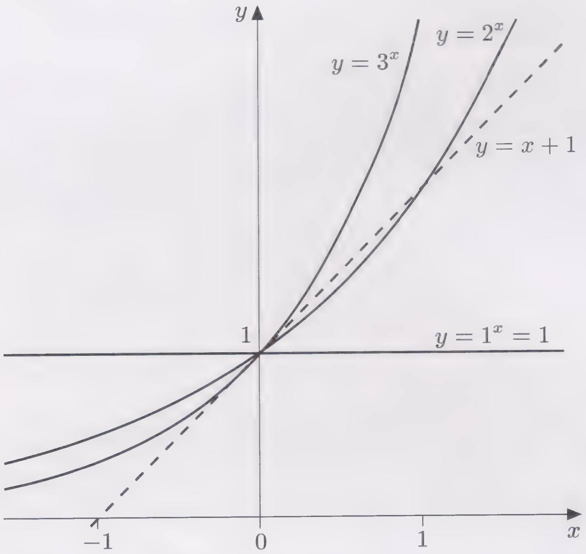


Figure 1.6 Graphs of exponential functions

The graphs all pass through the point $(0, 1)$, since $a^0 = 1$ whenever $a \neq 0$. The broken line that represents the graph of $y = x + 1$ also passes through the point $(0, 1)$.

It appears from Figure 1.6 that the greater the value of a, the steeper the graph of $y = a^x$ at the point $(0, 1)$. By comparison with the graph of $y = x + 1$, which has gradient 1 everywhere, it can be seen that the gradient of the graph of $y = 2^x$ at $(0, 1)$ appears to be *less* than 1, whereas that of $y = 3^x$ appears to be *greater* than 1. This observation suggests that for some a between 2 and 3, the gradient of the graph of $y = a^x$ at $(0, 1)$ is exactly 1. The number with this property is called e and its value as a decimal is $2.718\,281\ldots$. Thus for the particular function $f(x) = e^x$, we have $f'(0) = 1$, so

$$\lim_{h \to 0} \left(\frac{e^h - 1}{h} \right) = f'(0) = 1. \tag{1.8}$$

The graph of the function $f(x) = e^x$ is shown in Figure 1.7.

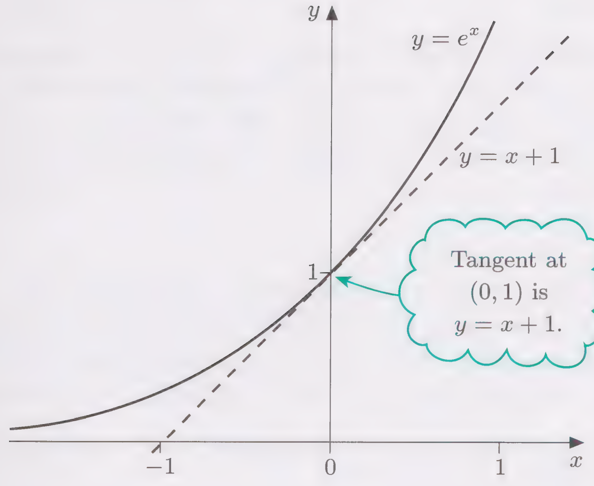


Figure 1.7 Graph of $f(x) = e^x$

The derivative of $f(x) = e^x$ can now be found at other points:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) \\ &= \lim_{h \to 0} \left(\frac{e^{x+h} - e^x}{h} \right) \\ &= \lim_{h \to 0} e^x \left(\frac{e^h - 1}{h} \right) \\ &= e^x \lim_{h \to 0} \left(\frac{e^h - 1}{h} \right) = e^x, \end{aligned}$$

By a rule for powers,
$$e^{x+h} = e^x e^h.$$

by equation (1.8). Thus we have obtained the following result.

> If $f(x) = e^x$, then $f'(x) = e^x$.

So the derived function of the exponential function $f(x) = e^x$ is itself! This is the reason why the number e is so important.

Last on our list of basic functions is $f(x) = \ln x$, the inverse function of the exponential function. For the moment we just state the derivative of this function, which will be obtained in Section 2 using a rule for differentiating inverse functions.

The term 'inverse function' was defined in MST121 Chapter A3, Subsection 4.1.

> If $f(x) = \ln x$, then $f'(x) = \dfrac{1}{x} \quad (x > 0)$.

Note that the domain of $f(x) = \ln x$ is $(0, \infty)$, and this interval is also the domain of f'.

Here is a list of the 'basic derivatives' introduced in this subsection.

A more extensive list is given in Section 2.

Table 1.1

Function $f(x)$	Derivative $f'(x)$
c	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x$	$1/x \quad (x > 0)$

Here n is a non-zero constant.

Higher derivatives

When we differentiate a smooth function f, we obtain the derived function f'. It is sometimes useful to differentiate the derived function f' to give a function f'' called the **second derived function**. The value of f'' at a given point is called the **second derivative** of f at that point. For example, if $f(x) = x^2$ then

We say 'f double-prime' or 'f double-dash'.

$$f'(x) = 2x, \quad \text{so} \quad f''(x) = 2.$$

If the differentiation process can be applied n times, then we obtain the **nth derived function**, which is denoted by $f^{(n)}$. The value $f^{(n)}(x)$ is called the **nth derivative** of f at x. If f can be differentiated n times at each point of its domain, then f is said to be **n-times differentiable**.

The notations f''' and $f^{(3)}$ are both used for the third derivative.

Such derivatives are called **higher derivatives** of f. The corresponding Leibniz notation for higher derivatives is

$$\frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \quad \text{and so on.}$$

We say 'dee-two-y by dee-x-squared', and so on.

Activity 1.3 Finding higher derivatives

(a) Given that $f(x) = e^x$, use Table 1.1 to write down

$$f'(x), \quad f''(x) \quad \text{and} \quad f^{(3)}(x).$$

(b) Given that $y = \sin x$, write down $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

Solutions are given on page 52.

Summary of Section 1

This section has introduced:

- ◇ differentiation of a function f to find the derivative $f'(x)$, which is the gradient of the tangent to the graph of f at the point $(x, f(x))$;
- ◇ standard notations used for differentiation;
- ◇ derivatives of several basic functions;
- ◇ higher derivatives, and the notations for these.

Exercises for Section 1

Exercise 1.1

Consider the function $f(x) = x^3 + 2x$.

(a) Show that if $h \neq 0$, then

$$\frac{f(x+h) - f(x)}{h} = 3x^2 + 3xh + h^2 + 2.$$

(b) Hence determine $f'(x)$.

Exercise 1.2

(a) Given that $g(t) = \ln t$, write down $g'(t)$ and $g''(t)$.

(b) Given that $s = t^3$, write down $\dfrac{ds}{dt}$, $\dfrac{d^2s}{dt^2}$, $\dfrac{d^3s}{dt^3}$ and $\dfrac{d^4s}{dt^4}$.

2 Differentiating combinations of functions

Starting from the known derivatives of the basic functions in Section 1, we can differentiate many more complicated functions, such as

$$f(x) = \sqrt{x^2 + 1}\cos(2x).$$

We differentiate such a function by analysing how $f(x)$ is built up from basic functions and then applying the appropriate differentiation rules.

2.1 Sum and Constant Multiple Rules

Sums and constant multiples of functions can be differentiated using the following rules.

Sum Rule

If k is a function with rule of the form $k(x) = f(x) + g(x)$, where f and g are smooth functions, then k is smooth and

$$k'(x) = f'(x) + g'(x).$$

Constant Multiple Rule

If k is a function with rule of the form $k(x) = cf(x)$, where f is a smooth function and c is a constant, then k is smooth and

$$k'(x) = cf'(x).$$

These two rules are often applied in combination. For example, if

$$k(x) = x^2 + 3\sin x,$$

then $k(x) = f(x) + cg(x)$, where $f(x) = x^2$, $g(x) = \sin x$ and $c = 3$. Since $f'(x) = 2x$ and $g'(x) = \cos x$, the Sum and Constant Multiple Rules give

$$k'(x) = 2x + 3\cos x.$$

These rules are usually applied without referring to them explicitly and without giving names to the constituent basic functions.

Activity 2.1 Using the Sum and Constant Multiple Rules

Differentiate each of the following functions.

(a) $k(x) = \dfrac{3}{x^4} + 2\ln x$ (b) $f(t) = 5e^t - \cos t$

Solutions are given on page 52.

2.2 Product and Quotient Rules

Products and quotients of functions can be differentiated using the following rules.

An outline proof of the Product Rule is given in Subsection 2.5.

Product Rule

If k is a function with rule of the form $k(x) = f(x)g(x)$, where f and g are smooth functions, then k is smooth and

$$k'(x) = f'(x)g(x) + f(x)g'(x). \tag{2.1}$$

Quotient Rule

If k is a function with rule of the form $k(x) = f(x)/g(x)$, where f and g are smooth functions, then k is smooth and

$$k'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}. \tag{2.2}$$

Note that if x is in the domain of k, then $g(x) \neq 0$. This condition follows from the domain convention.

For example, the function $k(x) = x^2 \sin x$ is the product of the basic functions $f(x) = x^2$ and $g(x) = \sin x$. Now $f'(x) = 2x$ and $g'(x) = \cos x$ so, by the Product Rule,

$$\begin{aligned} k'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= (2x)(\sin x) + (x^2)(\cos x) \\ &= 2x \sin x + x^2 \cos x. \end{aligned}$$

On the other hand, the function $k(x) = x^2/\sin x$ is the quotient of the functions $f(x) = x^2$ and $g(x) = \sin x$. Thus, by the Quotient Rule,

Here the domains of k and k' are both the set of numbers x for which $\sin x \neq 0$.

$$\begin{aligned} k'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \\ &= \frac{(\sin x)(2x) - (x^2)(\cos x)}{(\sin x)^2} = \frac{2x \sin x - x^2 \cos x}{\sin^2 x}. \end{aligned}$$

Although the Product Rule and Quotient Rule are more complicated than the Sum and Constant Multiple Rules, you should aim to be able to use them without assigning names to the constituent functions. Also, although it is not essential to make reference to these rules when you are using them, you may find it helpful to do so.

You may find the following versions of equations (2.1) and (2.2), in Leibniz notation, easier to work with.

Product Rule (Leibniz form)

If $y = uv$, where $u = f(x)$ and $v = g(x)$, then

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}.$$

Quotient Rule (Leibniz form)

If $y = u/v$, where $u = f(x)$ and $v = g(x) \neq 0$, then

$$\frac{dy}{dx} = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right).$$

These rules are sometimes written in the forms

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

and

$$\frac{dy}{dx} = \frac{\left(v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)}{v^2}.$$

The notation $\dfrac{d}{dx}(\quad)$ is especially useful when applying these rules. For example, for the function $f(x) = (\sin x)/x$, we take $u = \sin x$ and $v = x$, to obtain

$$f'(x) = \frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x\dfrac{d}{dx}(\sin x) - \sin x\dfrac{d}{dx}(x)}{x^2}$$
$$= \frac{x\cos x - \sin x}{x^2}.$$

Activity 2.2 Using the Product and Quotient Rules

Differentiate each of the following functions using the Product Rule or Quotient Rule, as appropriate.

(a) $k(x) = (x^2 - 1)(x^2 + x + 2)$ (b) $k(x) = \dfrac{x^2 - 1}{x^2 + x + 2}$

(c) $f(x) = e^x \cos x$ (d) $g(t) = \dfrac{e^t}{\cos t}$

Try to answer parts (c) and (d) without assigning names to the constituent functions.

Solutions are given on page 52.

Comment

In part (a), an alternative method is first to multiply out the product and then differentiate the resulting quartic polynomial.

The solutions indicate that after applying the Product Rule or Quotient Rule, we may need to simplify the resulting expression; for example, we often collect like terms, take out common factors and cancel.

The Quotient Rule can be used to establish several basic derivatives. For example, in Section 1 it was shown that for the function $f(x) = x^n$, where n is a positive integer, we have $f'(x) = nx^{n-1}$. A proof of this rule for the case of negative integers can now be given.

Suppose that $f(x) = x^n$, where n is a negative integer. Then $n = -m$, where m is a positive integer, so

$$f(x) = x^n = x^{-m} = \frac{1}{x^m}.$$

By the Quotient Rule,

$$f'(x) = \frac{x^m \times 0 - 1 \times mx^{m-1}}{(x^m)^2}$$
$$= -\frac{mx^{m-1}}{x^{2m}}.$$

Now $x^{m-1}/x^{2m} = x^{m-1-2m} = x^{-m-1}$, so

$$f'(x) = -mx^{-m-1} = nx^{n-1} \quad (\text{since } n = -m),$$

as required.

We can also obtain the derivatives of the functions tan, cot, sec and cosec, since these trigonometric functions have rules that are quotients:

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \operatorname{cosec} x = \frac{1}{\sin x}.$$

Activity 2.3 Derivatives of trigonometric functions

Use the Quotient Rule to establish each of the following derivatives.

(a) $\dfrac{d}{dx}(\tan x) = \sec^2 x$ (b) $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(c) $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

You are asked to establish the derivative of the function cosec in Exercise 2.2(a).

Solutions are given on page 53.

To differentiate some functions we have to use more than one rule, or make repeated use of the same rule. For example, the function

$$f(x) = x^4 \sin x \cos x$$

is a 'triple product'. To differentiate the function f, we first treat it as a product of *two* functions, such as

$$f(x) = (x^4 \sin x) \cos x.$$

We obtain, by the Product Rule,

$$f'(x) = \frac{d}{dx}(x^4 \sin x) \cos x + (x^4 \sin x)\frac{d}{dx}(\cos x).$$

Now $g(x) = x^4 \sin x$ is also a product, so a further application of the Product Rule gives

$$\begin{aligned} f'(x) &= (4x^3 \sin x + x^4 \cos x) \cos x + x^4 \sin x(-\sin x) \\ &= x^4(\cos^2 x - \sin^2 x) + 4x^3 \sin x \cos x. \end{aligned}$$

Now try some 'mixed' functions for yourself.

Activity 2.4 Differentiating mixed products and quotients

In Activity 2.3(a) you found that

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

Differentiate each of the following functions.

(a) $f(x) = x^4 e^x \sin x$ (b) $f(x) = \dfrac{x \tan x}{1 + x^2}$

Solutions are given on page 53.

2.3 Composite Rule

Read $g(f(x))$ as 'g of f of x'.

As you saw in Chapter B1, a function k with a rule of the form $k(x) = g(f(x))$, where g and f are functions, is called a composite function, known informally as a 'function of a function'. To differentiate such functions, we use the following rule.

> **Composite Rule**
>
> If k is a function with rule of the form $k(x) = g(f(x))$, where f and g are smooth functions, then k is smooth and
>
> $$k'(x) = g'(f(x))f'(x). \qquad (2.3)$$

For example, the function $k(x) = \sin(x^2)$ is of the form $k(x) = g(f(x))$, where the 'inner' function is $f(x) = x^2$ and the 'outer' function is the sine function. When first learning to use the Composite Rule, it is helpful to use an intermediate variable, u say, to label the output of the inner function $f(x)$; that is, we write

$$k(x) = g(f(x)) = g(u), \quad \text{where } u = f(x).$$

For the example $k(x) = \sin(x^2)$, we have

$$k(x) = g(u) = \sin u, \quad \text{where } u = f(x) = x^2.$$

Now

$$g'(u) = \cos u \quad \text{and} \quad f'(x) = 2x,$$

so, by the Composite Rule,

$$\begin{aligned} k'(x) &= g'(f(x))f'(x) \\ &= g'(u)f'(x) \\ &= (\cos u)(2x) \\ &= \cos(x^2) \times 2x \\ &= 2x\cos(x^2). \end{aligned}$$

As you become more familiar with applying the Composite Rule, you should find that you can write down the derivative of a simple composite function by first differentiating the outer function as if its argument were a single variable and then multiplying by the derivative of the inner function.

The Leibniz form of the Composite Rule is particularly memorable. In this form the result is often called the *Chain Rule.*

Chain Rule (Leibniz form of Composite Rule)

If $y = g(u)$, where $u = f(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Although this equation *looks like* a statement about fractions, recall that the three symbols for the derivatives are complete symbols.

For example, to differentiate the function $k(x) = \sin(x^2)$ using the Chain Rule, we decompose $y = \sin(x^2)$ as

$$y = \sin u, \quad \text{where } u = x^2.$$

Then

$$\frac{dy}{du} = \cos u \quad \text{and} \quad \frac{du}{dx} = 2x,$$

so, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\cos u)(2x) = 2x\cos(x^2),$$

as above.

Activity 2.5 Using the Composite Rule

Differentiate each of the following functions.

(a) $k(x) = e^{1/x}$ (b) $k(x) = \cos(3x)$

(c) $f(x) = \sqrt{\sin x}$ (d) $g(t) = \ln(-2t)$

Solutions are given on page 53.

Try to answer parts (c) and (d) without assigning names to the constituent functions.

Comment

◇ The Chain Rule can also be used to differentiate each of the above functions, and you may prefer this approach. For example, for part (a) we write $y = k(x) = e^{1/x}$ as

$$y = e^u, \quad \text{where } u = 1/x.$$

Then

$$\frac{dy}{du} = e^u \quad \text{and} \quad \frac{du}{dx} = -\frac{1}{x^2},$$

so, by the Chain Rule,

$$k'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\left(-\frac{1}{x^2}\right) = -\frac{e^{1/x}}{x^2}.$$

◇ Parts (b) and (d) illustrate how the Composite Rule enables us to generalise somewhat the basic derivatives given in Section 1, to cope with functions such as $\sin(ax), \cos(ax), e^{ax}$ and $\ln(ax)$, where a is any non-zero constant. For suppose that the derivative of a function g is known, and that the function k is defined by $k(x) = g(ax)$. Then we can apply the Composite Rule with inner function $f(x) = ax$ to obtain

$$k'(x) = g'(ax)f'(x) = ag'(ax),$$

since $f'(x) = a$.

This result gives

$$\frac{d}{dx}(\sin(ax)) = a\cos(ax), \quad \frac{d}{dx}(\cos(ax)) = -a\sin(ax),$$
$$\frac{d}{dx}(e^{ax}) = ae^{ax} \quad \text{and} \quad \frac{d}{dx}(\ln(ax)) = a\frac{1}{ax} = \frac{1}{x}.$$

Note that the derivative of $\ln(ax)$ does not involve a.

The Composite Rule can be used to establish another basic derivative. Earlier, we saw that the rule

$$\text{if } f(x) = x^n, \text{ then } f'(x) = nx^{n-1}$$

is true when n is a non-zero integer. This rule can now be established when n is a real number that is not an integer. In order to do this, we use the fact that $e^{\ln x} = x$, for $x > 0$, to write:

Recall that $\ln x$ is the unique number y such that $e^y = x$, so $e^{\ln x} = x$, for $x > 0$. See MST121 Chapter A3, Section 4.

$$f(x) = x^n = (e^{\ln x})^n = e^{n\ln x}, \quad \text{for } x > 0.$$

Then, by the Composite Rule,

$$f'(x) = e^{n\ln x}\left(\frac{n}{x}\right) = x^n\left(\frac{n}{x}\right) = nx^{n-1}, \quad \text{for } x > 0,$$

as required.

In the next activity, you are asked to differentiate functions whose rules involve a composite and also a product or quotient. For example, the function

$$f(x) = \frac{\cos(x^2)}{x}$$

is a quotient, and the 'top' function is a composite function.

We obtain

$$f'(x) = \frac{x\dfrac{d}{dx}(\cos(x^2)) - \cos(x^2)\dfrac{d}{dx}(x)}{x^2} \qquad \text{(Quotient Rule)}$$

$$= \frac{x(-2x\sin(x^2)) - \cos(x^2)\times 1}{x^2} \qquad \text{(Composite Rule)}$$

$$= \frac{-2x^2\sin(x^2) - \cos(x^2)}{x^2}$$

$$= -2\sin(x^2) - \frac{1}{x^2}\cos(x^2).$$

The simplification in this final line is not essential.

It is important to analyse such mixed functions carefully before applying the rules. For example,

- ◇ $k(x) = e^{x\sin x}$ involves a composite and a product, whereas
- ◇ $k(x) = e^x \sin x$ involves just a product.

Activity 2.6 Further mixed functions

Analyse and then differentiate each of the following functions.

(a) $f(x) = xe^{x^2}$ (b) $f(x) = \dfrac{\sin(\sqrt{x})}{\sqrt{x}}$

(c) $f(x) = \cos((x+4)\sec x)$ (d) $f(x) = \sqrt{x^2+1}\cos(2x)$

In Activity 2.3(c) you found that

$$\frac{d}{dx}(\sec x) = \sec x \tan x.$$

Solutions are given on page 54.

Comment

The solution to part (c) illustrates the fact that when differentiating a function whose rule involves an expression of the form $x + a$, where a is a constant, no extra term arises from differentiating $x + a$, because $\dfrac{d}{dx}(x+a) = 1$. For example, if $f(x) = \sin(x+2)$, then $f'(x) = \cos(x+2)$.

2.4 Inverse Rule

In Section 1 you met the basic derivatives

$$\frac{d}{dx}(e^x) = e^x \quad \text{and} \quad \frac{d}{dx}(\ln x) = \frac{1}{x}. \qquad (2.4)$$

The functions $f(x) = e^x$ and $g(x) = \ln x$ are closely related, since they are inverse functions of each other. This means that each function 'undoes' the effect of the other, in the sense that

See MST121 Chapter A3, Subsection 4.3.

$$f(g(x)) = e^{\ln x} = x \quad \text{and} \quad g(f(x)) = \ln(e^x) = x;$$

we write $f^{-1} = g$ and $g^{-1} = f$.

It is likely that there is a close connection between the derivatives of any pair of inverse functions, but none is immediately apparent in equations (2.4). We can establish such a connection for *any* pair of functions f and g that are inverses of each other by using the identity

$$f(g(x)) = x. \tag{2.5}$$

By the Composite Rule,

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x),$$

so differentiating the functions in equation (2.5) gives

$$f'(g(x))g'(x) = \frac{d}{dx}(x) = 1.$$

Thus we obtain the following connection between the derivatives of a pair of inverse functions.

Inverse Rule

If g is a function with rule of the form $g(x) = f^{-1}(x)$, where f is a smooth function, then

$$g'(x) = (f^{-1})'(x) = \frac{1}{f'(g(x))}, \quad \text{provided that } f'(g(x)) \neq 0. \tag{2.6}$$

Once again, there is a memorable Leibniz version of this rule, which is usually more convenient to use. It can be expressed as follows.

Inverse Rule (Leibniz form)

If $y = g(x)$ where $g = f^{-1}$, so $x = f(y)$, then

$$\frac{dy}{dx} = 1\Big/\frac{dx}{dy}, \quad \text{provided that } \frac{dx}{dy} \neq 0. \tag{2.7}$$

Here $\frac{dy}{dx}$ is the derivative of the function $g = f^{-1}$ at x, and $\frac{dx}{dy}$ is the derivative of the function f at the corresponding value $y = g(x)$.

Recall that the domain of the function ln is the set of positive real numbers.

Let us see how the Inverse Rule works for the function $g(x) = \ln x$, which is of the form $g(x) = f^{-1}(x)$ where $f(x) = e^x$. If $y = g(x) = \ln x$, then $x = e^y$, so $\frac{dx}{dy} = e^y$. Thus equation (2.7) gives

Note that, for all y we have

$$x = e^y \neq 0.$$

$$\frac{dy}{dx} = 1\Big/\frac{dx}{dy} = \frac{1}{e^y} = \frac{1}{x},$$

so

$$g'(x) = \frac{d}{dx}(\ln x) = \frac{1}{x},$$

The domain of the derivative of ln is the same as the domain of ln.

as in equations (2.4). In this calculation, it was straightforward to obtain the expression $\frac{dx}{dy}$ in terms of x, but often this requires more work, as in the following example.

Example 2.1 *Using the Inverse Rule*

Sketch the graph of the function $g(x) = \arcsin x$, and find $g'(x)$.

Solution

Since g is the inverse function of $f(x) = \sin x \ \left(x \in \left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]\right)$, its graph is as follows.

See MST121 Chapter A3, Subsection 4.2.

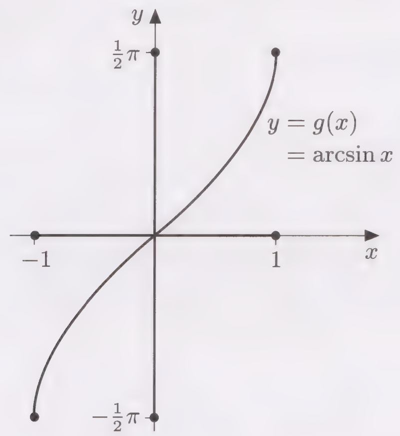


Figure 2.1 Graph of $g(x) = \arcsin x$

If $y = g(x) = \arcsin x$, then $x = \sin y$, so $\dfrac{dx}{dy} = \cos y$. Thus, by the Inverse Rule,

$$g'(x) = \frac{dy}{dx} = 1\Big/\frac{dx}{dy} = \frac{1}{\cos y}, \quad \text{provided that } \cos y \neq 0.$$

To express $\cos y$ in terms of x, we use $x = \sin y$ and the identity $\cos^2 y + \sin^2 y = 1$. We obtain

$$\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}.$$

Now the $+$ sign must be chosen here because $\cos y$ is never negative in the interval $\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$, where y lies. Thus

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}, \quad \text{provided that } x \neq \pm 1,$$

so

$$g'(x) = \frac{1}{\sqrt{1 - x^2}}, \quad \text{for } -1 < x < 1. \tag{2.8}$$

Comment

- ◇ It is possible to check that this answer is reasonable by comparing it with Figure 2.1. When $x = 0$, equation (2.8) gives the value 1, which agrees with the gradient at $(0, 0)$ on the graph. As x increases from 0 towards 1, the value of $1/\sqrt{1 - x^2}$ increases and becomes unbounded as x approaches 1. On the graph, you can see that the curve and hence the tangent become steeper, approaching vertical near the endpoint $\left(1, \frac{1}{2}\pi\right)$.
- ◇ It follows from equation (2.8) that the domain of g' is the open interval $(-1, 1)$, whereas the domain of g is the closed interval $[-1, 1]$, as shown in Figure 2.1.

Here are two further inverse functions for you to differentiate.

Activity 2.7 Using the Inverse Rule

For each of the following functions g, sketch the graph of g and find $g'(x)$.

(a) $g(x) = \arccos x$, the inverse function of $f(x) = \cos x$ $(x \in [0, \pi])$.

(b) $g(x) = \arctan x$, the inverse function of $f(x) = \tan x$ $(x \in (-\frac{1}{2}\pi, \frac{1}{2}\pi))$.

(*Hint*: In part (b) you can use the identity $1 + \tan^2\theta = \sec^2\theta$.)

See Chapter A3, Subsection 3.1.

Solutions are given on page 54.

Here is a list of the basic derivatives with which you need to be familiar.

The non-zero constant a does not appear in all cases, to help keep the table simple. The cases that feature a are obtained from entries in Table 1.1 by use of the Composite Rule, as in Activity 2.5(b).

Table 2.1

Function $f(x)$	Derivative $f'(x)$
c	0
x^n	nx^{n-1}
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
e^{ax}	ae^{ax}
$\ln(ax)$	$\dfrac{1}{x}$ $(ax > 0)$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$ $(-1 < x < 1)$
$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$ $(-1 < x < 1)$
$\arctan x$	$\dfrac{1}{1+x^2}$

Here $n \neq 0$.

The condition $ax > 0$ means that the domain of the derivative of the function $f(x) = \ln(ax)$ is

$$\begin{cases} (0, \infty), & \text{if } a > 0, \\ (-\infty, 0), & \text{if } a < 0. \end{cases}$$

These basic derivatives can be used in combination with the various rules, given earlier, to differentiate a wide range of functions. For example, to find the derivative of

$$f(x) = \arctan(x^3 + 1),$$

we write

$$f(x) = \arctan u, \quad \text{where } u = x^3 + 1.$$

Then, by the Chain Rule,

$$\begin{aligned} f'(x) &= \frac{d}{du}(\arctan u)\frac{du}{dx} \\ &= \frac{1}{1+u^2} \times 3x^2 \\ &= \frac{3x^2}{1+(x^3+1)^2} = \frac{3x^2}{x^6 + 2x^3 + 2}. \end{aligned}$$

Here are two similar types of function for you to differentiate.

Activity 2.8 Further composite functions

(a) Find the derivative of the function $f(x) = \arcsin(x^3)$.

(b) Find the first and second derivatives of

$$f(x) = \arctan(4x).$$

Solutions are given on page 55.

The final activity in this section asks you to differentiate some other functions, which are fairly common but are not on the list of basic derivatives.

Activity 2.9 Other common derivatives

(a) Find the derivatives of

$$f(x) = a^x \quad \text{and} \quad f^{-1}(x) = \log_a x,$$

where a is any positive real number other than 1.
(*Hint*: You can use the identity $a^x = e^{x \ln a}$.)

The function $\log_a$ was defined in MST121 Chapter A3, Subsection 4.3.

(b) Find the derivative of the function

$$f(x) = \ln|x| \quad (x \neq 0).$$

(*Hint*: You can use the fact that $\ln|x| = \ln x$, for $x > 0$, and $\ln|x| = \ln(-x)$, for $x < 0$.)

Solutions are given on page 55.

Comment

The derivative obtained in the solution of part (b) will play an important role in Chapter C2.

2.5 Proving the differentiation rules

This subsection will not be assessed.

In this subsection an outline proof is given of the Product Rule. Similar proofs can be given of the other rules that have been introduced in this section.

Suppose that k is a function with rule of the form $k(x) = f(x)g(x)$, where f and g are smooth functions. We need to evaluate

$$k'(x) = \lim_{h \to 0} \left(\frac{k(x+h) - k(x)}{h} \right).$$

Now

$$\frac{k(x+h)-k(x)}{h}=\frac{f(x+h)g(x+h)-f(x)g(x)}{h}.$$

We modify the right-hand side to include the expressions

$$\frac{f(x+h)-f(x)}{h} \quad \text{and} \quad \frac{g(x+h)-g(x)}{h}$$

because the rule for $k'(x)$ involves $f'(x)$ and $g'(x)$.

To do this, we add and subtract $f(x)g(x+h)$ in the numerator:

$$\begin{aligned}\frac{k(x+h)-k(x)}{h}&=\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}\\&=\left(\frac{f(x+h)-f(x)}{h}\right)g(x+h)+f(x)\left(\frac{g(x+h)-g(x)}{h}\right).\end{aligned}$$

As $h \to 0$, we have $g(x+h) \to g(x)$ and the two quotients tend to $f'(x)$ and $g'(x)$, respectively. Thus

$$k'(x)=\lim_{h\to 0}\left(\frac{k(x+h)-k(x)}{h}\right)=f'(x)g(x)+f(x)g'(x),$$

as required for the Product Rule.

Summary of Section 2

This section has introduced:

◇ the Sum Rule, Constant Multiple Rule, Product Rule, Quotient Rule, Composite Rule (Chain Rule) and Inverse Rule for differentiating functions that are defined in terms of other more basic functions;

◇ a table of basic derivatives;

◇ an outline proof of the Product Rule.

Exercises for Section 2

Exercise 2.1

Differentiate each of the following functions.

(a) $f(x)=2\ln x-3\arcsin x$ (b) $f(x)=x^3\tan x$ (c) $g(t)=\dfrac{t^2}{\cos(2t)}$

(d) $f(x)=\sin(1/x)$ (e) $f(x)=\cos^2(3x)$

(f) $g(y)=\dfrac{\ln(\cos y)}{y}$ $(0<y<\frac{1}{2}\pi)$ (g) $k(t)=\sin(t^2)\cos t$

(h) $f(x)=\arctan(e^{x^2})$ (i) $h(t)=2\sin(t+1)-t\cos(t^2-1)$

Exercise 2.2

(a) Use the Quotient Rule to establish the following derivative.

$$\frac{d}{dx}(\operatorname{cosec} x)=-\operatorname{cosec} x\cot x$$

(b) Use the Composite Rule to establish the following derivative.

$$\frac{d}{dx}\left(\arcsin\left(\frac{x}{a}\right)\right)=\frac{1}{\sqrt{a^2-x^2}} \quad (-a<x<a),$$

where a is a positive constant.

3 Graph sketching

To study Subsection 3.2 you will need an audio CD player and CDA5494.

In this section differentiation is included as one element of a general strategy for sketching the graphs of many functions. The aim of sketching the graph of a function is to provide a visual summary of the main properties of the function. Consider, for example, the function

$$f(x) = \frac{1}{1-x^2}.$$

A sketch of the graph of this function is given in Figure 3.1.

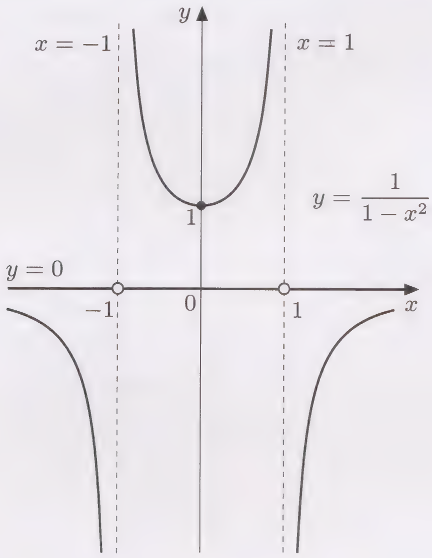


Figure 3.1 Graph of $y = 1/(1-x^2)$

Several key properties of the function f can be seen as features of the graph in Figure 3.1.

1. The domain of f consists of the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$.
2. The graph is symmetric under reflection in the y-axis.
3. The function f has y-intercept 1 and no x-intercepts.
4. The function f takes positive values on $(-1, 1)$, and takes negative values on $(-\infty, -1)$ and $(1, \infty)$.
5. The function f is increasing on $(0, 1)$ and $(1, \infty)$, and decreasing on $(-\infty, -1)$ and $(-1, 0)$.
6. The function f has vertical asymptotes $x = 1$ and $x = -1$, and horizontal asymptote $y = 0$.

Our graph-sketching strategy aims to show properties of these types.

An asymptote is a line which a curve approaches (arbitrarily closely) far from the origin; see Chapter A2, Subsection 2.2.

3.1 Determining features of a graph

In this subsection the various possible features of a graph are discussed in turn before combining these into a strategy. Activities involving the strategy are given in Subsection 3.2.

Domain

See Chapter B1, Subsection 1.1.

When the domain of a function is not specified, the convention is used that the domain is the largest possible set of real numbers for which the rule is applicable.

'Symmetry' features

If the graph of a function f is unchanged under reflection in the y-axis, as in Figure 3.2(a), then f is said to be an *even* function. Thus f is **even** if

$$f(-x) = f(x), \quad \text{for all } x \text{ in the domain of } f.$$

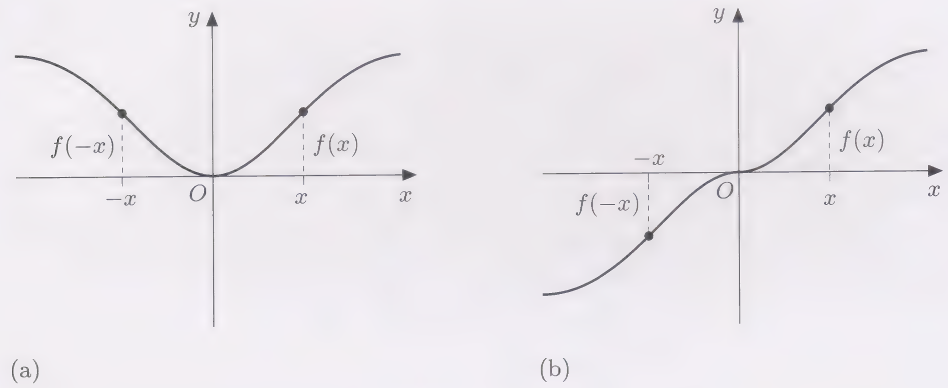


Figure 3.2 (a) An even function (b) An odd function

If the graph of f is unchanged by rotation through the angle π about the origin, as in Figure 3.2(b), then f is said to be an *odd* function. Thus f is **odd** if

The function $f(x) = 0$ is both odd and even.

$$f(-x) = -f(x), \quad \text{for all } x \text{ in the domain of } f.$$

For example:

◇ $f(x) = x^2$ is an even function, since

$$f(-x) = (-x)^2 = x^2 = f(x), \quad \text{for } x \in \mathbb{R};$$

Recall from Chapter B1 that the symbol $\in$ may be read as 'in' or 'belongs to', depending on the context.

◇ $f(x) = \sin x$ is an odd function, since

$$f(-x) = \sin(-x) = -\sin x = -f(x), \quad \text{for } x \in \mathbb{R};$$

◇ $f(x) = e^x$ is neither even nor odd, since we can find a value of x, say $x = 1$, such that $e^{-x} \neq \pm e^x$.

We have $e^{-1} = 0.367\ldots$ and $e^1 = 2.718\ldots$.

Intercepts

An **intercept** is a value of x or y at which a graph $y = f(x)$ of a function f meets the x- or y-axis, respectively. The x-intercepts are the solutions (if any) of the equation $f(x) = 0$, also known as the **zeros** of f. The y-intercept is the value $f(0)$; see Figure 3.3.

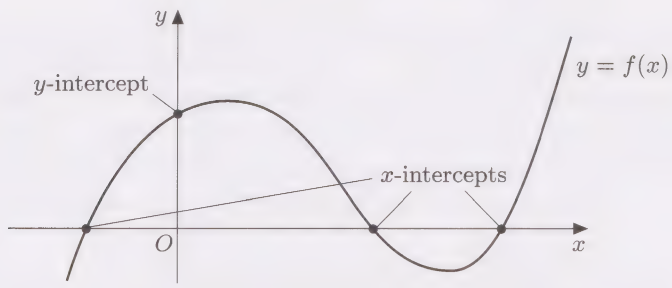


Figure 3.3 Intercepts

It is usually straightforward to calculate the y-intercept $f(0)$, but for some functions f the x-intercepts are difficult to find, since the equation $f(x) = 0$ must be solved. In cases where this is difficult, it is useful to locate approximately the zeros of the function f by finding intervals in which the values of f change sign. For example, consider the function $f(x) = x^3 - 3x - 3$. Since $f(2) = -1$ and $f(3) = 15$, f has a zero in the interval $(2, 3)$.

Intervals of constant sign

Closely related to the x-intercepts of a function are the intervals on which the function has constant sign. We say that f is **positive** on an interval I if

$$f(x) > 0, \quad \text{for all } x \in I,$$

and that f is **negative** on I if

$$f(x) < 0, \quad \text{for all } x \in I.$$

Sometimes we can find such intervals by factorisation and use of a 'sign table'. For example, we can express

$$f(x) = \frac{1}{1 - x^2} \quad \text{as} \quad f(x) = \frac{1}{(1-x)(1+x)},$$

and thus construct a sign table for $f(x)$, as below. In the left-hand column of the table are the factors that appear in $f(x)$. In the top row are the key values of x at which the factors are zero and the intervals on either side of these key values. The signs of the various factors are shown in the table by $+, -$ or 0, from which the sign of $f(x)$ may be deduced.

x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
$1 - x$	$+$	$+$	$+$	0	$-$
$1 + x$	$-$	0	$+$	$+$	$+$
$f(x)$	$-$	$*$	$+$	$*$	$-$

The symbol $*$ indicates a point not in the domain of f.

From the bottom row, we deduce that f is positive on $(-1, 1)$, and negative on $(-\infty, -1)$ and $(1, \infty)$, as shown in Figure 3.1.

Intervals on which a function is increasing or decreasing

One aim of graph sketching is to indicate the intervals on which a function is either increasing or decreasing. We can sometimes determine these intervals by using just the rule of the function. For example, the function $f(x) = x^2$ is increasing on $(0, \infty)$ because

The concept of increasing or decreasing on an interval was introduced in Chapter B1, Subsection 2.1.

$$\text{if } 0 < x_1 < x_2, \text{ then } 0 < x_1^2 < x_2^2.$$

For a smooth function f, we can use the derived function f' to identify such intervals. Since the derivative $f'(x)$ gives the gradient at the point $(x, f(x))$ on the graph of f, we can use the value of $f'(x)$ to gain information about the shape of the graph near this point. If $f'(x) = 0$, then the tangent to the graph at $(x, f(x))$ is horizontal, and x is called a **stationary point** of f; see Figure 3.4, where x_1, x_2 and x_3 are stationary points.

The name 'stationary point' is often used to refer also to the corresponding point $(x, f(x))$ on the graph of f.

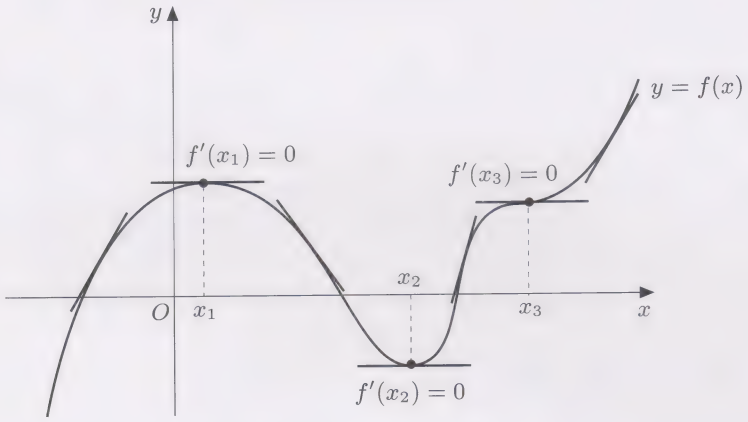


Figure 3.4 Three stationary points

On those intervals in the domain of f which include no stationary points, the function f' has constant sign and we can deduce that f is either increasing or decreasing on that interval.

> **Increasing/Decreasing Criterion**
> Let I be an open interval in the domain of a smooth function f.
> - ◇ If $f'(x) > 0$ for all x in I, then f is increasing on I.
> - ◇ If $f'(x) < 0$ for all x in I, then f is decreasing on I.

For example, in Figure 3.4:

- ◇ f is increasing on each of the intervals $(-\infty, x_1)$, (x_2, x_3) and (x_3, ∞) because $f'(x) > 0$ for x in these intervals;
- ◇ f is decreasing on the interval (x_1, x_2) because $f'(x) < 0$ for x in this interval.

In fact, f is increasing on (x_2, ∞), but this does not follow immediately from the Increasing/Decreasing Criterion because $f'(x_2) = 0$.

The function f in Figure 3.4 has a **local maximum** at x_1, with value $f(x_1)$, because $f(x_1)$ is the greatest function value in the immediate vicinity of x_1. More precisely, there is some open interval I containing x_1 such that $f(x) \leq f(x_1)$, for all $x \in I$. Similarly, f has a **local minimum** at x_2, with value $f(x_2)$.

Local maxima and local minima of smooth functions always occur at stationary points, but a stationary point does not have to be a local maximum or local minimum. For example, in Figure 3.4 the function f has neither a local maximum nor a local minimum at the stationary point x_3.

The plural of 'maximum' is 'maxima' and the plural of 'minimum' is 'minima'.

As you can check $x = 0$ is a stationary point of the function $f(x) = x^3$, but this point is neither a local maximum nor a local minimum of f.

The following test can be used to discover whether a function f has a local maximum or local minimum at a known stationary point.

First Derivative Test

Suppose that x_0 is a stationary point of a smooth function f; that is, $f'(x_0) = 0$.

◇ If $f'(x)$ changes sign from positive to negative as x increases through x_0, then f has a local maximum at x_0.

◇ If $f'(x)$ changes sign from negative to positive as x increases through x_0, then f has a local minimum at x_0.

◇ If $f'(x)$ does not change sign as x increases through x_0, then f has neither a local maximum nor a local minimum at x_0.

Figure 3.5 illustrates why the First Derivative Test works.

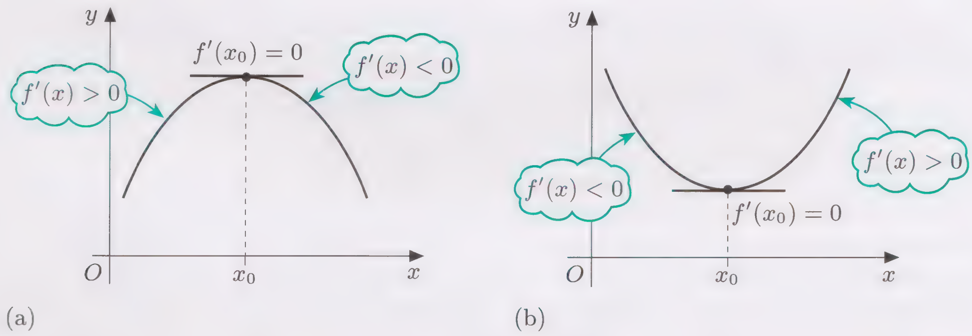


Figure 3.5 (a) Local maximum (b) Local minimum

To determine whether the derivative of a smooth function f changes sign as x increases through a stationary point x_0, it is often sufficient to check the sign of the derivative at 'test points' to the left and the right of x_0. Alternatively, we can use a sign table for $f'(x)$. The latter approach has the advantage that it helps us to understand the behaviour of f near any vertical asymptotes, as you will see.

The procedure based on 'test points' was described in MST121 Chapter C1.

For example, for the function $f(x) = 1/(1 - x^2)$ in Figure 3.1, we have

$$f'(x) = \frac{2x}{(1-x^2)^2}.$$

A sign table for $f'(x)$ is as follows.

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \infty)$
$2x$	$-$	$-$	$-$	0	$+$	$+$	$+$
$(1-x^2)^2$	$+$	0	$+$	$+$	$+$	0	$+$
$f'(x)$	$-$	$*$	$-$	0	$+$	$*$	$+$

In this case there is no need to factorise $(1 - x^2)^2$, since

$$(1 - x^2)^2 \geq 0.$$

Thus f has a stationary point at 0. Also, by the Increasing/Decreasing Criterion, f is increasing on $(0, 1)$ and $(1, \infty)$, and decreasing on $(-\infty, -1)$ and $(-1, 0)$. In particular, by the First Derivative Test, f has a local minimum at 0.

Asymptotic behaviour

For a function f, the term **asymptotic behaviour** refers to the behaviour of points on the graph of $y = f(x)$ for which the variable x or the variable y take arbitrarily large values. Thus describing the asymptotic behaviour includes finding any asymptotes of the function and also stating occurrences of any of the types of behaviour illustrated in Figure 3.6.

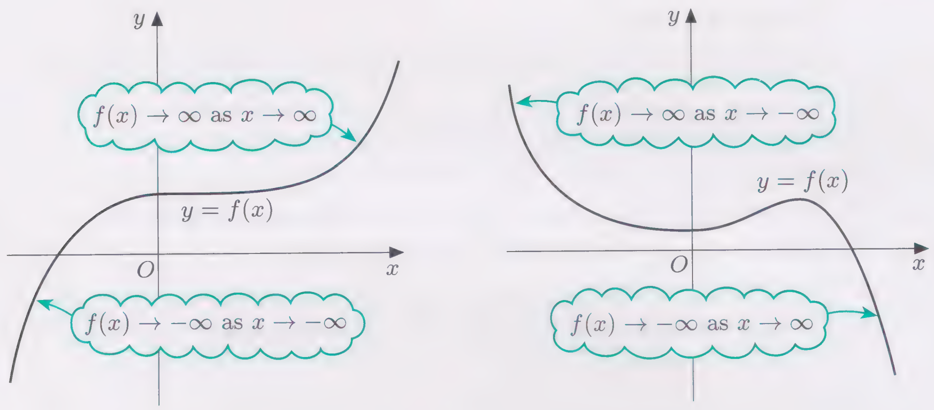


Figure 3.6 Four types of asymptotic behaviour

We now consider the asymptotic behaviour of polynomial functions and rational functions.

The asymptotic behaviour of a polynomial function of degree n,

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $n \geq 1$ and $a_n \neq 0$, is similar to that of the function $g(x) = a_n x^n$ because the term $a_n x^n$ dominates the other terms for large x. We call x^n the **dominant term** and a_n the **leading coefficient** of $f(x)$. This asymptotic behaviour is described in Table 3.1, for $a_n > 0$.

If $a_n < 0$, then

$$f(x) \to \infty$$

is replaced by

$$f(x) \to -\infty,$$

and vice versa throughout this table.

Table 3.1 Asymptotic behaviour of f, where $a_n > 0$

	$x \to \infty$	$x \to -\infty$
n even	$f(x) \to \infty$	$f(x) \to \infty$
n odd	$f(x) \to \infty$	$f(x) \to -\infty$

For example, if

$$f(x) = 2x^4 - 4x^3 + 1,$$

then, since the leading coefficient 2 is positive and the power of the dominant term x^4 is even, we have

$$f(x) \to \infty \text{ as } x \to \infty \quad \text{and} \quad f(x) \to \infty \text{ as } x \to -\infty.$$

A **rational function** is a function with rule of the form

$$f(x) = \frac{p(x)}{q(x)},$$

It is assumed that $p(x)$ and $q(x)$ have no common factors.

where both p and q are polynomial functions. The domain of f is $\mathbb{R}$ except for the points where $q(x) = 0$. For example, $f(x) = 1/(1 - x^2)$ is a rational function, with $p(x) = 1$ and $q(x) = 1 - x^2$. Since the zeros of q are 1 and -1, the domain of f is $\mathbb{R}$ except for these points.

The rational functions we consider can have two types of asymptote:

- ◇ a **vertical asymptote** has equation of the form $x = a$;
- ◇ a **horizontal asymptote** has equation of the form $y = b$.

These two types of asymptote are illustrated in Figure 3.7.

A rational function can also have a *slant asymptote*, which has equation of the form $y = mx + c$, where $m \neq 0$. In this chapter we do not consider such asymptotes.

Asymptotes are usually indicated by broken lines, except when they coincide with an axis.

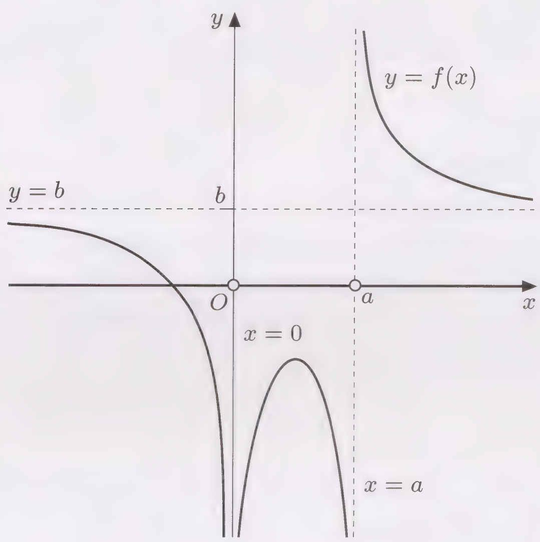


Figure 3.7 Vertical and horizontal asymptotes

A vertical asymptote of a rational function $f(x) = p(x)/q(x)$ occurs at those points a for which $q(a) = 0$ and $p(a) \neq 0$. For example, the rational function $f(x) = 1/(1 - x^2)$ has vertical asymptotes $x = 1$ and $x = -1$; see Figure 3.1.

More generally, a function f of the form $f(x) = h(x)/q(x)$, where h is a continuous function and q is a polynomial function, has an asymptote $x = a$ if $q(a) = 0$ and $h(a) \neq 0$. You will see an example of such a function in Activity 3.1.

To detect horizontal asymptotes of a rational function, we divide both the numerator and the denominator by the dominant term of the *denominator*, and consider the behaviour as $x \to \infty$ and as $x \to -\infty$. For example, if $f(x) = 1/(1 - x^2)$, then x^2 is the dominant term of the denominator. Hence

$$f(x) = \frac{1}{1 - x^2} = \frac{1/x^2}{1/x^2 - 1}$$

$$\to \frac{0}{0 - 1} = 0 \text{ as } x \to \pm\infty,$$

so f has the horizontal asymptote $y = 0$; see Figure 3.1.

The expression 'as $x \to \pm\infty$' is shorthand for 'as $x \to \infty$ and as $x \to -\infty$'.

3.2 Graph-sketching strategy

The following strategy summarises the main features that a sketch of the graph of a function should convey.

> **Strategy**
> To sketch the graph of a given function f, determine the following features of f (where possible) and then show these features in your sketch.
>
> 1. The domain of f.
> 2. Whether f is even or odd.
> 3. The x- and y-intercepts of f.
> 4. The intervals on which f is positive or negative.
> 5. The intervals on which f is increasing or decreasing and any stationary points, local maxima and local minima.
> 6. The asymptotic behaviour of f.

Remarks

1. For some graphs it may not be necessary or possible to find all the features listed in the strategy.
2. Choose the scales on your axes with care. Ideally the two scales should be the same, but sometimes unequal scales are needed to show some features clearly.
3. The intercepts of f and any local maxima and minima serve to locate parts of the graph of a function. It may be necessary to locate other parts by evaluating a few points on the curve. In part (a) of Activity 3.1 some points are suggested for this purpose.

Now listen to CDA5494 (Tracks 1–6), band 1, 'Graph sketching'.

Frame 1

Sketch the graph of $f(x) = \dfrac{2x-3}{x-1}$

Step 1 Domain of f: $(-\infty, 1)$ and $(1, \infty)$

Step 2 Even/odd: f is neither.

$f(2) = 1,\ f(-2) = \frac{7}{3} \neq \pm f(2)$

Step 3 Intercepts: $f(x) = 0$ when $2x - 3 = 0$, so x-intercept is $\frac{3}{2}$.
$f(0) = -3/-1 = 3$, so y-intercept is 3.

Step 4 Positive/negative:

x	$(-\infty, 1)$	1	$\left(1, \frac{3}{2}\right)$	$\frac{3}{2}$	$\left(\frac{3}{2}, \infty\right)$
$2x-3$	$-$	$-$	$-$	0	$+$
$x-1$	$-$	0	$+$	$+$	$+$
$f(x)$	$+$	$*$	$-$	0	$+$

f is positive on $(-\infty, 1)$ and $\left(\frac{3}{2}, \infty\right)$, negative on $\left(1, \frac{3}{2}\right)$.

Step 5 Increasing/decreasing:

$$f'(x) = \frac{(x-1)2 - (2x-3)1}{(x-1)^2} = \frac{1}{(x-1)^2}$$

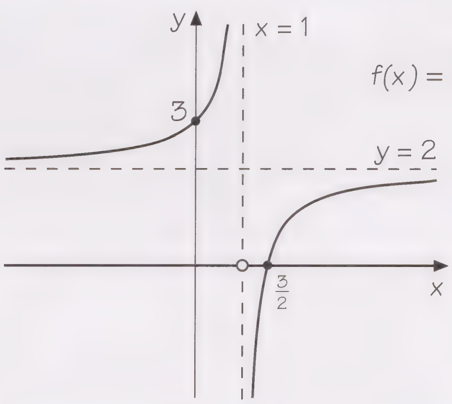
x	$(-\infty, 1)$	1	$(1, \infty)$
$(x-1)^2$	$+$	0	$+$
$f'(x)$	$+$	$*$	$+$

f is increasing on $(-\infty, 1)$ and $(1, \infty)$.
f has no stationary points.

Step 6 Asymptotic behaviour: $x = 1$ is a vertical asymptote.

$$f(x) = \frac{2x-3}{x-1} = \frac{2-3/x}{1-1/x} \to \frac{2-0}{1-0} = 2 \text{ as } x \to \pm\infty,$$

so $y = 2$ is a horizontal asymptote.



$f(x) = \dfrac{2x-3}{x-1}$

Frame 2

Sketch the graph of $f(x) = \dfrac{x}{x^2+1}$

Step 1 Domain of f: $\mathbb{R}$

Step 2 Even/odd: $f(-x) = \dfrac{-x}{(-x)^2+1} = -\dfrac{x}{x^2+1} = -f(x)$, so f is odd.

Step 3 Intercepts: $f(x) = 0$ when $x = 0$, so only intercept is origin.

Step 4 Positive/negative:

x	$(-\infty, 0)$	0	$(0, \infty)$
x	$-$	0	$+$
x^2+1	$+$	1	$+$
$f(x)$	$-$	0	$+$

f is positive on $(0, \infty)$, negative on $(-\infty, 0)$.

Step 5 Increasing/decreasing:

$$f'(x) = \frac{(x^2+1)1 - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2}$$

x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
$1-x$	$+$	$+$	$+$	0	$-$
$1+x$	$-$	0	$+$	$+$	$+$
$(x^2+1)^2$	$+$	$+$	$+$	$+$	$+$
$f'(x)$	$-$	0	$+$	0	$-$

f is increasing on $\boxed{-1, 1}$, decreasing on $\boxed{-\infty, -1}$ and $\boxed{1, \infty}$.

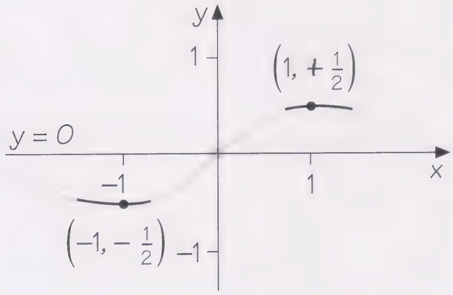
f has stationary points ± 1, local maximum at $\boxed{1}$, value $\boxed{\tfrac{1}{2}}$,

local minimum at $\boxed{-1}$, value $\boxed{-\tfrac{1}{2}}$.

Step 6 Asymptotic behaviour: f has no vertical asymptotes.

$$f(x) = \frac{x}{x^2+1} = \frac{1/x}{1+1/x^2} \to \frac{0}{1+0} = 0 \text{ as } x \to \pm\infty,$$

so $y = 0$ is a horizontal asymptote.



Frame 3

Sketch the graph of $f(x) = (x-2)\,e^x$

You may assume: $f(x) \to 0$ as $x \to -\infty$.

Step 1 Domain of f: $\mathbb{R}$

Step 2 Even/odd: f is neither.

$f(1) = -e$, $f(-1) = -3e^{-1} \neq \pm f(1)$

Step 3 Intercepts: $f(x) = 0$ when $x - 2 = 0$, so x-intercept is 2.

$f(0) = -2$, so y-intercept is -2.

Step 4 Positive/negative:

x	$(-\infty, 2)$	2	$(2, \infty)$
$x-2$	$-$	0	$+$
e^x	$+$	$+$	$+$
$f(x)$	$-$	0	$+$

f is positive on $(2, \infty)$, negative on $(-\infty, 2)$.

Step 5 Increasing/decreasing:

$$\begin{aligned} f'(x) &= (x-2)\,e^x + e^x \times 1 \\ &= (x-1)\,e^x \end{aligned}$$

x	$(-\infty, 1)$	1	$(1, \infty)$
$x-1$	$-$	0	$+$
e^x	$+$	$+$	$+$
$f'(x)$	$-$	0	$+$

f is increasing on $(1, \infty)$, decreasing on $(-\infty, 1)$.

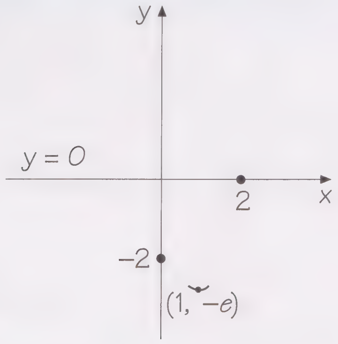
f has stationary point 1, a local minimum, value $f(1) = -e$.

Step 6 Asymptotic behaviour: f has no vertical asymptotes.

$f(x) \to 0$ as $x \to -\infty$ (given),

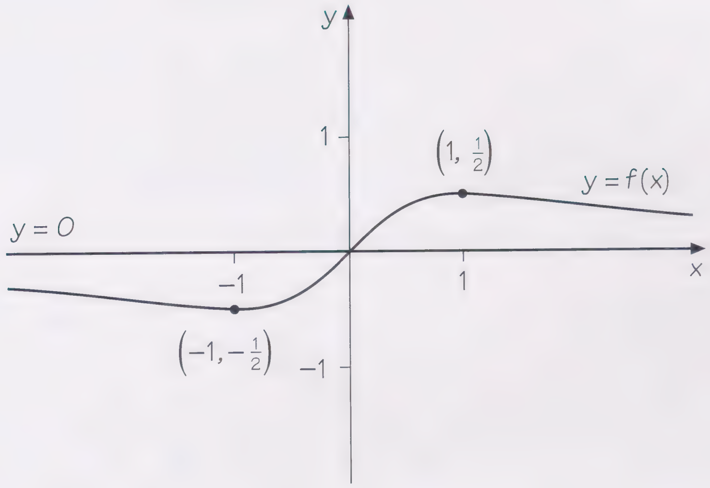
so $y = 0$ is a horizontal asymptote for the left part of the graph;

$f(x) \to \infty$ as $x \to \infty$ (since $e^x \to \infty$ as $x \to \infty$).

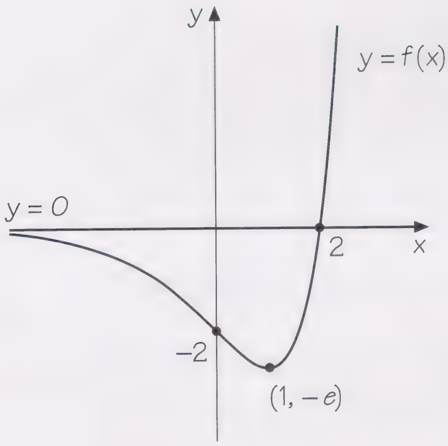


Frame 4

Graph of $f(x) = \dfrac{x}{x^2+1}$



Graph of $f(x) = (x-2)\,e^x$



Activity 3.1 Using the graph-sketching strategy

Sketch the graph of each of the following functions.

(a) $f(x) = \dfrac{4x+1}{3x-5}$

(b) $f(x) = \dfrac{3x}{x^2-4}$

(c) $f(x) = \dfrac{e^x}{x}$ (You may assume that $e^x/x \to \infty$ as $x \to \infty$.)

(d) $f(x) = x^3 - 3x - 3$

Solutions are given on page 55.

In part (a) it will help to consider the points for which $x = 3$ and $x = 4$.

In part (d), you need not find accurate values for the x-intercepts.

Comment

The assumption given in part (c) about the behaviour of e^x/x as x tends to ∞ is a particular case of the property that, as x tends to infinity, e^x tends to infinity faster than any power of x. To be precise, for each fixed positive integer n, we have

$$\frac{e^x}{x^n} \to \infty \text{ as } x \to \infty.$$

Using the second derivative

In Subsection 3.1 the First Derivative Test was introduced to determine whether a given stationary point gives a local maximum, a local minimum, or neither. There is an alternative test for a local maximum and a local minimum, using the second derivative.

See MST121 Chapter C1, Subsection 2.2.

Second Derivative Test

Suppose that x_0 is a stationary point of a smooth function f; that is, $f'(x_0) = 0$.

- ◇ If $f''(x_0) < 0$, then f has a local maximum at x_0.
- ◇ If $f''(x_0) > 0$, then f has a local minimum at x_0.

Strictly speaking, f is required to be a twice-differentiable function.

This test can be very efficient as a means of classifying stationary points. However, for some functions it is complicated to find the second derivative. Moreover, if $f''(x_0) = 0$, then the Second Derivative Test gives no result; the stationary point may be a local maximum, a local minimum or neither in this case. This is why the First Derivative Test has been used here.

Summary of Section 3

This section has introduced:

- ◇ some features of the graphs of functions, which can be identified without differentiation, including symmetry properties and intervals on which the function is positive or negative;
- ◇ some features of the graphs of functions, which can be identified with the help of differentiation, including intervals on which the function is increasing or decreasing, stationary points, local maxima and local minima, and asymptotic behaviour;
- ◇ a strategy for sketching graphs of smooth functions.

Exercises for Section 3

Exercise 3.1

Sketch the graph of each of the following functions.

(a) $f(x) = \dfrac{2-x}{x+3}$ (You will find it convenient to use unequal scales for this graph.)

(b) $f(x) = \dfrac{x^2}{x^2+5}$

(c) $f(x) = \dfrac{e^{3x}}{(x+1)^2}$ (You may assume that $f(x) \to \infty$ as $x \to \infty$.)

(d) $f(x) = x^3 - 4x$

4 Newton–Raphson method

Solving equations is one of the most important activities in mathematics because many problems where we seek to find an 'unknown' number x lead to an equation, or equations, involving x. Most equations cannot be solved exactly, so we must be content with approximate solutions. The *Newton–Raphson method* is one of the most effective techniques for finding approximate solutions of equations rapidly. Special cases of this method were used by the Ancient Babylonians, but the method is named after Isaac Newton and Joseph Raphson who gave versions of the method suitable for solving polynomial equations. In its usual modern formulation, the method seems to be due to Thomas Simpson.

Raphson (1648–1715) and Simpson (1710–1761) were English mathematicians. Raphson published his version of the method in 1690. Simpson's textbook *Treatise of Algebra* was widely used during the eighteenth century.

4.1 What is the Newton–Raphson method?

Suppose that f is a smooth function, and that we want to find a zero of f; that is, a solution of the equation

$$f(x) = 0.$$

In this situation it is often possible to find a number which is *near* to a solution of the equation. For example, we can calculate the values of the function at various points in order to find an interval in which the function values change sign, as you will see.

Suppose that, as in Figure 4.1, the equation $f(x) = 0$ has a solution a, and we know that the number x_0 is near a.

One way of obtaining the number x_0 is to examine the graph of f.

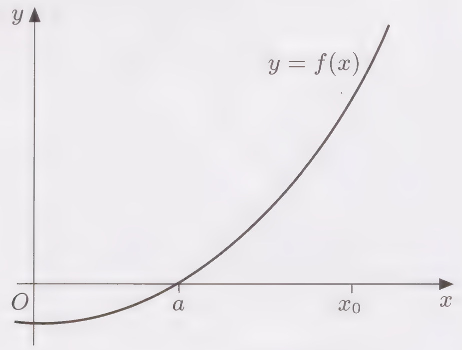


Figure 4.1 Initial approximate solution x_0

One way to calculate a number that is usually closer to the solution a than x_0 is to determine the point, x_1 say, where the tangent to the graph of $y = f(x)$ at $(x_0, f(x_0))$ cuts the x-axis; see Figure 4.2 overleaf.

For some functions f and some numbers x_0 it is possible that x_1 is further from a than x_0 is.

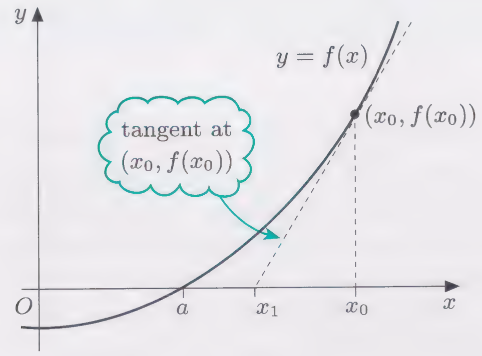


Figure 4.2 Second approximate solution x_1

For this tangent,

$$\text{rise} = f(x_0)$$

and

$$\text{run} = x_0 - x_1.$$

To find the point x_1, we note that the gradient of this tangent is $f'(x_0)$, so

$$f'(x_0) = \frac{f(x_0)}{x_0 - x_1}.$$

On solving this equation for x_1, we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Having found a value x_1 that is nearer to a than x_0, we can repeat the process using the corresponding formula

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

to obtain a value x_2 that is usually even closer to a, as in Figure 4.3.

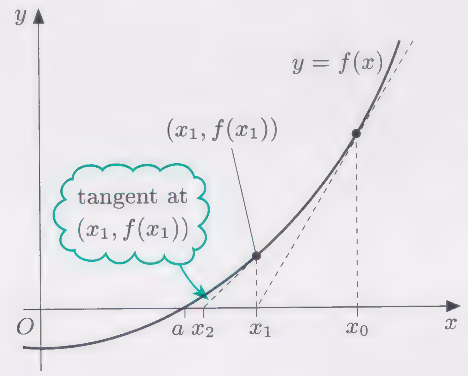


Figure 4.3 Third approximate solution x_2

Repetition of this process usually gives better and better approximations to a solution of the equation $f(x) = 0$. This iterative process is called the *Newton–Raphson method*.

Some discussion of the convergence of this method is given in Subsection 4.2.

Newton–Raphson method

Let f be a smooth function. The Newton–Raphson method for finding an approximate solution of the equation $f(x) = 0$ is to start with an initial term x_0 and calculate the iteration sequence given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n = 0, 1, 2, \ldots). \tag{4.1}$$

In the following activity you are asked to apply the Newton–Raphson method to a particular cubic function.

Activity 4.1 Using the Newton–Raphson method

Consider the function $f(x) = x^3 - 2x - 2$.

(a) By evaluating $f(1)$ and $f(2)$, show that the equation $f(x) = 0$ has a solution in the interval $(1, 2)$.

(b) Show that for this function f the Newton–Raphson formula in equation (4.1) can be expressed as

$$x_{n+1} = \frac{2x_n^3 + 2}{3x_n^2 - 2} \quad (n = 0, 1, 2, \ldots).$$

(c) Use your calculator to find the first five terms of the sequence x_n when the initial term is $x_0 = 2$. (Ideally you should use the maximum accuracy of your calculator and make appropriate use of its memory facilities.)

(d) Check that the final term calculated in part (c) is a good approximation to a solution of $f(x) = 0$.

Solutions are given on page 58.

Comment

In part (a) we used the fact that, for any polynomial function f, if $f(x_1)$ and $f(x_2)$ have opposite signs, then f has a zero at some point between x_1 and x_2.

This method of locating a zero also applies to any function f that is continuous on a closed interval which contains x_1 and x_2.

The Newton–Raphson method gives an excellent iterative method of calculating square roots by using the fact that if $a > 0$, then $\sqrt{a}$ is a solution of the equation $x^2 - a = 0$.

Activity 4.2 Calculating square roots

Suppose that $a > 0$. Show that the Newton–Raphson formula (4.1) applied to the equation

$$x^2 - a = 0$$

gives the iteration sequence

$$x_{n+1} = \tfrac{1}{2}\left(x_n + \frac{a}{x_n}\right) \quad (n = 0, 1, 2, \ldots).$$

The function associated with this equation is

$$f(x) = x^2 - a.$$

A solution is given on page 58.

Comment

You used this formula for the case $a = 3$ in Chapter B1, Exercise 1.1(b).

4.2 Advantages and disadvantages of the Newton–Raphson method

In Activity 4.1 you found the following terms given by the Newton–Raphson method for the equation $f(x) = x^3 - 2x - 2 = 0$:

$$x_0 = 2,$$
$$x_1 = 1.8,$$
$$x_2 = 1.769\,948\,187,$$
$$x_3 = 1.769\,292\,663,$$
$$x_4 = 1.769\,292\,354.$$

You also found that the term x_4 is a good approximation to a solution of $f(x) = 0$, in the sense that $f(x_4) = 0$ (to 8 d.p.). It is striking that the method has produced such an accurate solution in only four iterations, but this rapid convergence is a feature of the Newton–Raphson method.

An explanation of the rapid convergence of the Newton–Raphson method can be given by applying the methods for dealing with iteration sequences described in Chapter B1. First we observe that equation (4.1) can be written in the form

$$x_{n+1} = g(x_n) \quad (n = 0, 1, 2, \ldots),$$

where $g(x) = x - f(x)/f'(x)$. If a is a solution of the equation $f(x) = 0$ and $f'(a) \neq 0$, then $f(a) = 0$ and

$$g(a) = a - \frac{f(a)}{f'(a)} = a.$$

Therefore a is a fixed point of the function g, but what type of fixed point is it: attracting, indifferent or repelling?

See Chapter B1, Subsection 2.2, for this classification of fixed points.

To classify the fixed point a we need to examine the value of $g'(a)$. We can differentiate g, using the Quotient Rule, to obtain

$$\begin{aligned} g'(x) &= 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2} \\ &= 1 - \frac{(f'(x))^2}{(f'(x))^2} + \frac{f(x)f''(x)}{(f'(x))^2} \\ &= \frac{f(x)f''(x)}{(f'(x))^2}. \end{aligned}$$

Since $f(a) = 0$ and $f'(a) \neq 0$, we deduce that

$$g'(a) = \frac{f(a)f''(a)}{(f'(a))^2} = 0.$$

Therefore the fixed point a is in fact a super-attracting fixed point of the function g. Hence it is to be expected that any iteration sequence generated by $g(x) = x - f(x)/f'(x)$, with initial term near a, converges extremely rapidly to a, which is a zero of f.

However, there are some problems that may occur when applying the Newton–Raphson method, as you will see in the next activity.

Activity 4.3 Failure of the Newton–Raphson method

Consider the function $f(x) = x^3 - 2x - 2$, for which the Newton–Raphson formula is

You met this function in Activity 4.1.

$$x_{n+1} = \frac{2x_n^3 + 2}{3x_n^2 - 2} \quad (n = 0, 1, 2, \ldots).$$

(a) Calculate x_1 when $x_0 = 0.8165$. What possible problem with the Newton–Raphson method does your calculation suggest?

(b) Calculate x_1 and x_2 when $x_0 = 0$. What possible problem with the Newton–Raphson method does your calculation suggest?

Solutions are given on page 58.

Comment

Parts (a) and (b) indicate two circumstances under which the Newton–Raphson method may *fail*; that is, the sequence generated does not converge to a zero of the function f. Another such circumstance is illustrated in Figure 4.4.

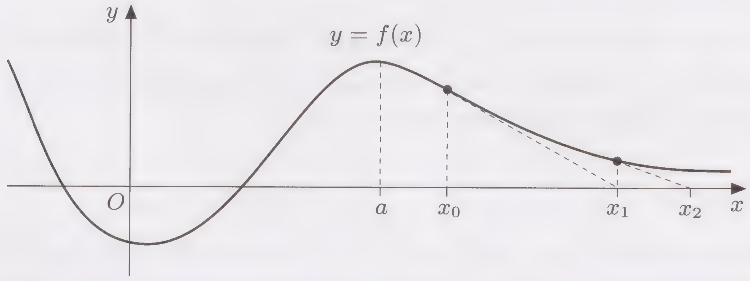


Figure 4.4 Newton–Raphson method failing to converge to a zero

In this case, if $x_0 > a$, then the sequence x_n given by the Newton–Raphson method tends to infinity.

Summary of Section 4

This section has introduced:

- ◇ the Newton–Raphson method for finding approximate solutions to equations of the form $f(x) = 0$, where f is a smooth function;
- ◇ an explanation of why the Newton–Raphson method is often very effective;
- ◇ some circumstances under which the Newton–Raphson method fails.

Exercises for Section 4

Exercise 4.1

You sketched the graph of this function in Activity 3.1(d).

Consider the function $f(x) = x^3 - 3x - 3$.

(a) Show that the equation $f(x) = 0$ has a solution in the interval $(2, 3)$.

(b) Show that for this function f the Newton–Raphson formula in equation (4.1) can be expressed as

$$x_{n+1} = \frac{2x_n^3 + 3}{3x_n^2 - 3} \quad (n = 0, 1, 2, \ldots).$$

(c) Use your calculator to find the first five terms of the sequence x_n when the initial term is $x_0 = 2$. (Use your calculator's maximum accuracy.)

(d) Check that the final term calculated in part (c) is a good approximation to a solution of $f(x) = 0$.

(e) Find a value of x_0 for which the Newton–Raphson method fails for this function f.

Exercise 4.2

An approximate solution of the equation $x - \cos x = 0$ was found in Chapter B1, Exercise 1.3(a), by a different method.

Consider the function $f(x) = x - \cos x$.

(a) Show that the equation $f(x) = 0$ has a solution in the interval $(0, \frac{1}{2}\pi)$.

(b) Show that for this function f the Newton–Raphson formula in equation (4.1) can be expressed as

$$x_{n+1} = \frac{x_n \sin(x_n) + \cos(x_n)}{1 + \sin(x_n)} \quad (n = 0, 1, 2, \ldots).$$

(c) Use your calculator to find the first five terms of the sequence x_n when the initial term is $x_0 = 0$. (Use your calculator's maximum accuracy.)

(d) Check that the final term calculated in part (c) is a good approximation to a solution of $f(x) = 0$.

(e) Find a value of x_0 for which the Newton–Raphson method fails for this function f.

5 Differentiation with the computer

To study this section you will need access to your computer, together with the Mathcad files for this chapter and Computer Book C.

In this section you will see that the computer can be used to find the derivative of any smooth function whose rule is entered. This means that, even where the function concerned is too complicated to permit differentiation 'by hand' with ease, an answer can be obtained from the computer.

You will also see that the Newton–Raphson method can be implemented on the computer. Using the computer, you will be able to explore the Newton–Raphson method, finding situations in which it successfully solves equations and others where it fails.

Refer to Computer Book C for work in this section.

Summary of Section 5

In this section you saw how the computer can be used to find derivatives, and also how it can be used to implement the Newton–Raphson method.

Summary of Chapter C1

In this chapter you met the process of differentiation of a function f to find the derivative, which is the gradient of the tangent to the graph of f at a point on the graph. By means of various rules, the process of differentiation can be applied to a wide range of smooth functions.

Differentiation can be used to help sketch the graphs of smooth functions and also to find approximate values for the zeros of such functions, by the Newton–Raphson method.

Learning outcomes

You have been working towards the following learning outcomes.

Terms to know and use

Calculus, gradient, tangent, differentiation, derivative, derived function, differentiable, smooth, power function, nth derivative, nth derived function, n-times-differentiable, even, odd, positive (or negative) on an interval, stationary point, local maximum, local minimum, asymptotic behaviour, dominant term, leading coefficient.

Notation to know and use

The various notations associated with derivatives:

- ◇ $f'(x)$, $f''(x)$, $f^{(3)}(x)$, ... (function notation);
- ◇ $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}, \ldots,$ and $\dfrac{d}{dx}(f(x))$ (Leibniz notation).

The notations for limits, as applied to the definition of the derivative:

- ◇ $f'(x) = \lim\limits_{h \to 0} \left(\dfrac{f(x+h) - f(x)}{h} \right)$ (function notation);
- ◇ $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$ (Leibniz notation).
- ◇ Other notation: $x \to \pm\infty$.

Mathematical skills

- ◇ Use the limit definition to calculate derivatives in simple cases.
- ◇ Differentiate any function that is expressed in terms of the basic functions using the Sum, Constant Multiple, Product, Quotient and Composite Rules.
- ◇ Apply the Inverse Rule to differentiate a function that can be expressed as the inverse of another function.
- ◇ Determine higher derivatives.
- ◇ Apply the graph-sketching strategy.
- ◇ Apply the Newton–Raphson method.

Mathcad skills

◇ Use Mathcad where appropriate to find derivatives (and higher derivatives) of complicated functions.

◇ Use Mathcad to experiment with the Newton–Raphson method.

Ideas to be aware of

◇ That in finding derivatives from the definition, various rules for limits are assumed.

◇ That the derivative of the sine function can be deduced from geometric reasoning.

◇ That the rules for differentiation are proved by using the definition of the derivative together with appropriate rearrangements and reasoning.

◇ That the Newton–Raphson method is often very effective, but it does not always give a solution.

Solutions to Activities

Solution 1.1

(a) We have

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{1}{h}\left(\frac{1}{(x+h)^2}-\frac{1}{x^2}\right)\\ &= \frac{1}{h}\left(\frac{x^2-(x+h)^2}{(x+h)^2x^2}\right)\\ &= \frac{-2xh-h^2}{h(x+h)^2x^2}\\ &= \frac{-2x-h}{(x+h)^2x^2},\end{aligned}$$

as required.

(b) By equation (1.2), we need to find the limit as h tends to 0 of the quotient from part (a):

$$\begin{aligned}\lim_{h\to 0}\left(\frac{-2x-h}{(x+h)^2x^2}\right) &= \frac{-2x}{x^2\times x^2}\\ &= -\frac{2}{x^3},\end{aligned}$$

so

$$f'(x) = -\frac{2}{x^3}.$$

(c) By part (b) we have

$$f'(2) = -\frac{2}{2^3} = -\tfrac{1}{4}$$

and

$$f'(-1) = -\frac{2}{(-1)^3} = 2.$$

The corresponding tangents at $\left(2,\frac{1}{4}\right)$ and $(-1,1)$, respectively, are sketched in Figure S.1.

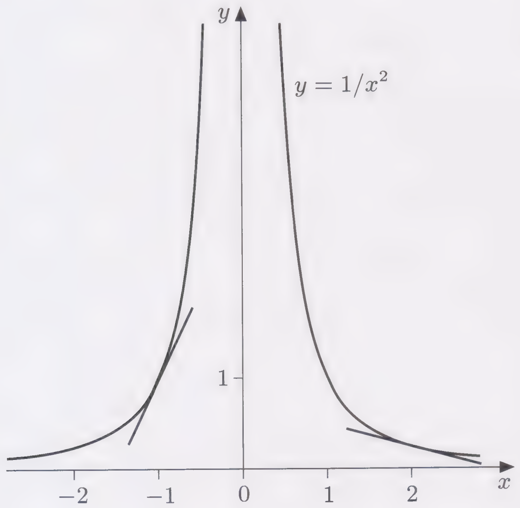


Figure S.1

Solution 1.2

(a) Here $f(x) = x^{100}$, so $f'(x) = 100x^{99}$.

(b) Since

$$f(x) = \frac{1}{x^{3/2}} = x^{-3/2},$$

we have

$$f'(x) = -\tfrac{3}{2}x^{-5/2} = -\frac{3}{2x^{5/2}}.$$

Solution 1.3

(a) Here $f(x) = e^x$, so

$$f'(x) = e^x,\ f''(x) = e^x \text{ and } f^{(3)}(x) = e^x.$$

(b) Here $y = \sin x$, so

$$\frac{dy}{dx} = \cos x \text{ and } \frac{d^2y}{dx^2} = -\sin x.$$

Solution 2.1

(a) Here $k(x) = 3x^{-4} + 2\ln x$, so

$$\begin{aligned}k'(x) &= 3(-4)x^{-5} + 2\left(\frac{1}{x}\right)\\ &= -\frac{12}{x^5} + \frac{2}{x}.\end{aligned}$$

(b) Here $f(t) = 5e^t - \cos t$, so

$$\begin{aligned}f'(t) &= 5e^t - (-\sin t)\\ &= 5e^t + \sin t.\end{aligned}$$

Solution 2.2

(a) In this case $k(x) = f(x)g(x)$, where $f(x) = x^2-1$ and $g(x) = x^2+x+2$. Since $f'(x) = 2x$ and $g'(x) = 2x+1$, the Product Rule gives

$$\begin{aligned}k'(x) &= (2x)(x^2+x+2) + (x^2-1)(2x+1)\\ &= (2x^3+2x^2+4x) + (2x^3+x^2-2x-1)\\ &= 4x^3+3x^2+2x-1.\end{aligned}$$

(b) In this case $k(x) = f(x)/g(x)$, where $f(x) = x^2-1$ and $g(x) = x^2+x+2$. Since $f'(x) = 2x$ and $g'(x) = 2x+1$, the Quotient Rule gives

$$\begin{aligned}k'(x) &= \frac{(x^2+x+2)(2x) - (x^2-1)(2x+1)}{(x^2+x+2)^2}\\ &= \frac{(2x^3+2x^2+4x) - (2x^3+x^2-2x-1)}{(x^2+x+2)^2}\\ &= \frac{x^2+6x+1}{(x^2+x+2)^2}.\end{aligned}$$

(c) Here $f(x) = e^x \cos x$, so by the Product Rule

$$f'(x) = \frac{d}{dx}(e^x)\cos x + e^x \frac{d}{dx}(\cos x)$$
$$= e^x \cos x + e^x(-\sin x)$$
$$= e^x(\cos x - \sin x).$$

(d) Here $g(t) = e^t/\cos t$, so by the Quotient Rule

$$g'(t) = \frac{\cos t \frac{d}{dt}(e^t) - e^t \frac{d}{dt}(\cos t)}{\cos^2 t}$$
$$= \frac{(\cos t)e^t - e^t(-\sin t)}{(\cos t)^2}$$
$$= \frac{e^t(\cos t + \sin t)}{\cos^2 t}.$$

Solution 2.3

(a) We have

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$
$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x}$$
$$= \sec^2 x,$$

as required.

(b) We have

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$
$$= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2}$$
$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x}$$
$$= -\frac{1}{\sin^2 x}$$
$$= -\operatorname{cosec}^2 x,$$

as required.

(c) We have

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$
$$= \frac{(\cos x)(0) - 1(-\sin x)}{(\cos x)^2}$$
$$= \frac{\sin x}{\cos^2 x}$$
$$= \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)$$
$$= \sec x \tan x,$$

as required.

Solution 2.4

(a) Since $f(x) = (x^4 e^x)\sin x$, we can apply the Product Rule twice to obtain

$$f'(x) = \frac{d}{dx}\left(x^4 e^x\right)\sin x + (x^4 e^x)\frac{d}{dx}(\sin x)$$
$$= \left(4x^3 e^x + x^4 e^x\right)\sin x + x^4 e^x \cos x$$
$$= x^3 e^x\left((4+x)\sin x + x\cos x\right).$$

(b) We apply the Quotient Rule and then the Product Rule:

$$f'(x) = \frac{(1+x^2)\frac{d}{dx}(x\tan x) - (x\tan x)\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$
$$= \frac{(1+x^2)(\tan x + x\sec^2 x) - (x\tan x)(2x)}{(1+x^2)^2}$$
$$= \frac{x(1+x^2)\sec^2 x + (1-x^2)\tan x}{(1+x^2)^2}.$$

Solution 2.5

(a) In this case $k(x) = g(f(x)) = g(u)$, where

$$g(u) = e^u \quad \text{and} \quad u = f(x) = 1/x.$$

Since

$$g'(u) = e^u \quad \text{and} \quad f'(x) = -1/x^2,$$

we have

$$k'(x) = e^u \times \left(-\frac{1}{x^2}\right) = e^{1/x} \times \left(-\frac{1}{x^2}\right) = -\frac{e^{1/x}}{x^2}.$$

(b) In this case $k(x) = g(f(x)) = g(u)$, where

$$g(u) = \cos u \quad \text{and} \quad u = f(x) = 3x.$$

Since

$$g'(u) = -\sin u \quad \text{and} \quad f'(x) = 3,$$

we have

$$k'(x) = (-\sin u) \times 3 = -3\sin(3x).$$

(c) Since $f(x) = \sqrt{\sin x} = (\sin x)^{1/2}$, we have

$$f'(x) = \tfrac{1}{2}(\sin x)^{-1/2} \times \cos x = \frac{\cos x}{2\sqrt{\sin x}}.$$

(d) Here $g(t) = \ln(-2t)$, so

$$g'(t) = \left(\frac{1}{-2t}\right) \times (-2) = \frac{1}{t}.$$

Solution 2.6

(a) The rule $f(x) = xe^{x^2}$ involves a product and a composite, so we apply the Product Rule and then the Composite Rule:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x)e^{x^2} + x\frac{d}{dx}\left(e^{x^2}\right) \\ &= 1 \times e^{x^2} + x\left(e^{x^2} \times 2x\right) \\ &= (2x^2+1)e^{x^2}. \end{aligned}$$

(b) The rule $f(x) = \sin(\sqrt{x})/\sqrt{x}$ involves a quotient and a composite, so we apply the Quotient Rule and then the Composite Rule:

$$\begin{aligned} f'(x) &= \frac{\sqrt{x}\frac{d}{dx}(\sin(\sqrt{x})) - \sin(\sqrt{x})\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2} \\ &= \frac{\sqrt{x}\cos(\sqrt{x})\left(\frac{1}{2}x^{-1/2}\right) - \sin(\sqrt{x})\left(\frac{1}{2}x^{-1/2}\right)}{x} \\ &= \frac{\sqrt{x}\cos(\sqrt{x}) - \sin(\sqrt{x})}{2x^{3/2}}. \end{aligned}$$

(c) The rule $f(x) = \cos((x+4)\sec x)$ involves a composite and a product, so we apply the Composite Rule and then the Product Rule:

$$\begin{aligned} f'(x) &= -\sin((x+4)\sec x)\frac{d}{dx}((x+4)\sec x) \\ &= -\sin((x+4)\sec x) \\ &\quad \times \left(\frac{d}{dx}(x+4)\sec x + (x+4)\frac{d}{dx}(\sec x)\right) \\ &= -\sin((x+4)\sec x)(\sec x + (x+4)\sec x \tan x) \\ &= -\sec x(1+(x+4)\tan x)\sin((x+4)\sec x). \end{aligned}$$

(d) The rule $f(x) = \sqrt{x^2+1}\cos(2x)$ involves a product and a composite, so we apply the Product Rule and then the Composite Rule:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\sqrt{x^2+1}\right)\cos(2x) + \sqrt{x^2+1}\frac{d}{dx}(\cos(2x)) \\ &= \tfrac{1}{2}(x^2+1)^{-1/2}(2x)(\cos(2x)) \\ &\quad + \sqrt{x^2+1}(-2\sin(2x)) \\ &= \frac{x}{\sqrt{x^2+1}}\cos(2x) - 2\sqrt{x^2+1}\sin(2x). \end{aligned}$$

Solution 2.7

(a) Since g is the inverse function of $f(x) = \cos x$ $(x \in [0, \pi])$, its graph is as follows.

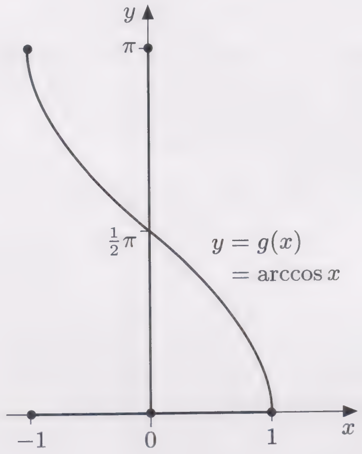


Figure S.2

If $y = \arccos x$, then $x = \cos y$, so $\dfrac{dx}{dy} = -\sin y$. Thus, by the Inverse Rule,

$$\frac{dy}{dx} = 1\Big/\frac{dx}{dy} = \frac{1}{-\sin y}.$$

Since $\cos^2 y + \sin^2 y = 1$, we obtain

$$\sin y = \pm\sqrt{1-\cos^2 y} = \sqrt{1-x^2},$$

because $\sin y \geq 0$ for $0 \leq y \leq \pi$. Thus

$$g'(x) = \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}, \qquad \text{for } -1 < x < 1.$$

(Note that the domain of g is the closed interval $[-1, 1]$, as indicated in Figure S.2, whereas the domain of g' is the open interval $(-1, 1)$.)

(b) Since g is the inverse function of $f(x) = \tan x$ $\left(x \in \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)\right)$, its graph is as follows.

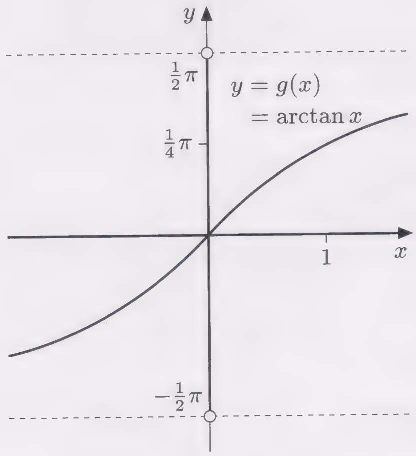


Figure S.3

If $y = \arctan x$, then $x = \tan y$, so $\dfrac{dx}{dy} = \sec^2 y$; see Activity 2.3(a). Thus, by the Inverse Rule,

$$\frac{dy}{dx} = 1\Big/\frac{dx}{dy} = \frac{1}{\sec^2 y}.$$

Since $\sec^2 y = 1 + \tan^2 y = 1 + x^2$, we obtain

$$g'(x) = \frac{dy}{dx} = \frac{1}{1+x^2}.$$

(Note that $\mathbb{R}$ is the domain of g and g'.)

Solution 2.8

Leibniz notation is used in part (a) but not in part (b).

(a) We can write

$$f(x) = \arcsin(u), \quad \text{where } u = x^3,$$

so, by the Chain Rule,

$$\begin{aligned} f'(x) &= \frac{d}{du}(\arcsin u)\frac{du}{dx} \\ &= \frac{1}{\sqrt{1-u^2}} \times 3x^2 \\ &= \frac{3x^2}{\sqrt{1-x^6}}, \quad \text{for } -1 < x < 1. \end{aligned}$$

(b) By the Composite Rule,

$$f'(x) = \frac{1}{1+(4x)^2} \times 4 = \frac{4}{1+16x^2}.$$

Thus, by the Composite Rule again (or the Quotient Rule),

$$\begin{aligned} f''(x) &= 4\left(\frac{-1}{(1+16x^2)^2}\right) \times (32x) \\ &= \frac{-128x}{(1+16x^2)^2}. \end{aligned}$$

Solution 2.9

(a) Since $f(x) = a^x = e^{x \ln a}$, the Composite Rule gives

$$\begin{aligned} f'(x) &= e^{x \ln a} \times \ln a \\ &= a^x \ln a. \end{aligned}$$

To find $(f^{-1})'(x)$, let $y = f^{-1}(x) = \log_a x$ so $x = f(y) = a^y$. Since

$$\frac{dx}{dy} = f'(y) = a^y \ln a,$$

as we have just seen, the Inverse Rule gives

$$\frac{dy}{dx} = 1\Big/\frac{dx}{dy} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a},$$

so

$$(f^{-1})'(x) = \frac{dy}{dx} = \frac{1}{x \ln a}.$$

(b) From Table 2.1,

$$\frac{d}{dx}(\ln(ax)) = \frac{1}{x}, \quad \text{for } ax > 0.$$

Thus, with $a = 1$,

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \quad \text{for } x > 0,$$

and, with $a = -1$,

$$\frac{d}{dx}(\ln(-x)) = \frac{1}{x}, \quad \text{for } x < 0.$$

We deduce that

$$f'(x) = \frac{1}{x} \quad (x \neq 0).$$

Solution 3.1

In each case your sketch should be fairly close to the accurate graph shown.

(a) $f(x) = \dfrac{4x+1}{3x-5}$.

Step 1: The denominator $3x - 5$ is 0 when $x = \frac{5}{3}$, so the domain of f is $\mathbb{R}$ except for $\frac{5}{3}$.

Step 2: The function f is neither even nor odd.

Step 3: The only x-intercept occurs when $4x + 1 = 0$; that is, $x = -\frac{1}{4}$. The y-intercept is $f(0) = -\frac{1}{5}$.

Step 4: Here is a sign table for $f(x)$.

x	$(-\infty, -\frac{1}{4})$	$-\frac{1}{4}$	$(-\frac{1}{4}, \frac{5}{3})$	$\frac{5}{3}$	$(\frac{5}{3}, \infty)$
$4x+1$	$-$	0	$+$	$+$	$+$
$3x-5$	$-$	$-$	$-$	0	$+$
$f(x)$	$+$	0	$-$	$*$	$+$

Thus f is positive on $(\frac{5}{3}, \infty)$ and $(-\infty, -\frac{1}{4})$, and negative on $(-\frac{1}{4}, \frac{5}{3})$.

Step 5: The derivative of f is

$$f'(x) = \frac{(3x-5)4 - (4x+1)3}{(3x-5)^2} = \frac{-23}{(3x-5)^2},$$

so $f'(x) < 0$ for all x in the domain of f. Thus f is decreasing on $(-\infty, \frac{5}{3})$ and $(\frac{5}{3}, \infty)$.

Step 6: Since the denominator is 0 when x is $\frac{5}{3}$, the line $x = \frac{5}{3}$ is a vertical asymptote. Also

$$\begin{aligned} f(x) &= \frac{4x+1}{3x-5} = \frac{4+1/x}{3-5/x} \\ &\to \frac{4+0}{3-0} = \tfrac{4}{3} \text{ as } x \to \pm\infty. \end{aligned}$$

So the line $y = \frac{4}{3}$ is a horizontal asymptote.

To locate the curve in the interval $(\frac{5}{3}, \infty)$, we calculate $f(3) = \frac{13}{4} = 3\frac{1}{4}$ and $f(4) = \frac{17}{7} = 2\frac{3}{7}$.

Thus we can sketch the following graph.

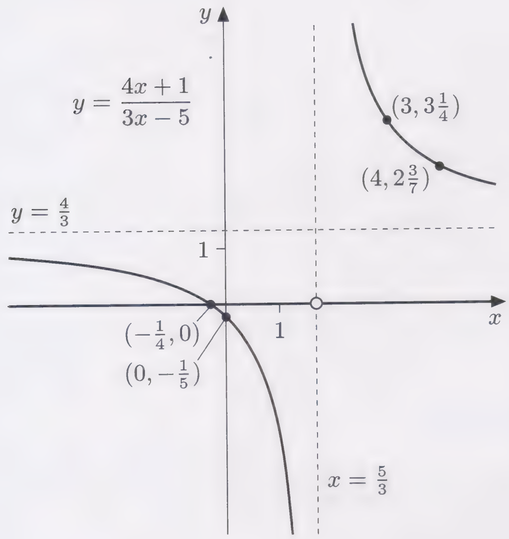


Figure S.4

(b) $f(x) = \dfrac{3x}{x^2-4}$

Step 1: The domain of f is $\mathbb{R}$ except for ± 2, where $x^2 - 4 = 0$.

Step 2: The function f is odd because

$$f(-x) = \frac{3(-x)}{(-x)^2-4} = -\frac{3x}{x^2-4} = -f(x).$$

Step 3: The only x-intercept occurs when $3x = 0$; that is, $x = 0$. The y-intercept is $f(0) = 0$.

Step 4: Here is a sign table for

$$f(x) = \frac{3x}{x^2-4} = \frac{3x}{(x-2)(x+2)}.$$

x	$(-\infty,-2)$	-2	$(-2,0)$	0	$(0,2)$	2	$(2,\infty)$
$3x$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x+2$	$-$	0	$+$	$+$	$+$	$+$	$+$
$f(x)$	$-$	$*$	$+$	0	$-$	$*$	$+$

Thus f is positive on $(-2, 0)$ and $(2, \infty)$, and negative on $(-\infty, -2)$ and $(0, 2)$.

Step 5: The derivative of f is

$$f'(x) = \frac{(x^2-4)3 - 3x(2x)}{(x^2-4)^2}$$
$$= -\frac{3x^2+12}{(x^2-4)^2},$$

so $f'(x) < 0$ for all x in the domain. Thus f is decreasing on $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$.

Step 6: Since the denominator is 0 when x is ± 2, the lines $x = 2$ and $x = -2$ are vertical asymptotes. Also

$$f(x) = \frac{3x}{x^2-4} = \frac{3/x}{1-4/x^2}$$
$$\to \frac{0}{1-0} = 0 \text{ as } x \to \pm\infty.$$

So the line $y = 0$ is a horizontal asymptote.

To locate the curve in the interval $(2, \infty)$, we again calculate $f(3) = \frac{9}{5} = 1\frac{4}{5}$ and $f(4) = 1$. To locate the curve in the interval $(-\infty, -2)$, we use the fact that f is odd.

Thus we can sketch the following graph.

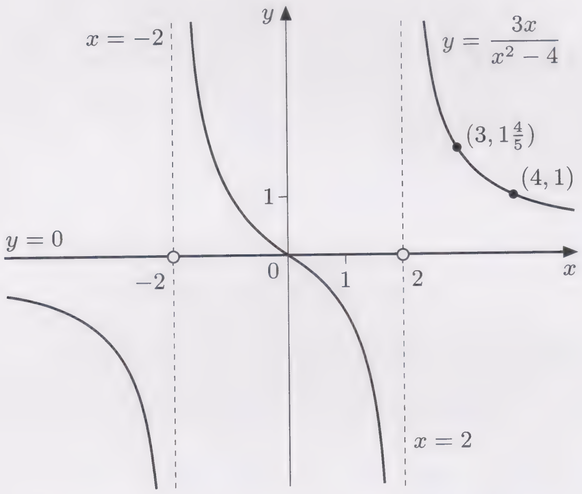


Figure S.5

(c) $f(x) = \dfrac{e^x}{x}$

Step 1: The domain of f is $\mathbb{R}$ except 0.

Step 2: The function f is neither even nor odd.

Step 3: Since $e^x \neq 0$, for $x \in \mathbb{R}$, there are no x-intercepts, and since 0 is not in the domain, there is no y-intercept.

Step 4: Here is a sign table for $f(x)$.

x	$(-\infty,0)$	0	$(0,\infty)$
e^x	$+$	$+$	$+$
x	$-$	0	$+$
$f(x)$	$-$	$*$	$+$

Thus f is positive on $(0, \infty)$ and negative on $(-\infty, 0)$.

Step 5: The derivative of f is

$$f'(x) = \frac{xe^x - e^x \times 1}{x^2} = \frac{e^x(x-1)}{x^2},$$

which has the following sign table.

x	$(-\infty,0)$	0	$(0,1)$	1	$(1,\infty)$
e^x	$+$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	0	$+$
x^2	$+$	0	$+$	$+$	$+$
$f'(x)$	$-$	$*$	$-$	0	$+$

Thus f is increasing on $(1,\infty)$, and decreasing on $(-\infty,0)$ and $(0,1)$. Also f has a stationary point at 1, which is a local minimum with value $f(1) = e$.

Step 6: The line $x = 0$ is a vertical asymptote. Also

$f(x) \to \infty$ as $x \to \infty$ (given)

$f(x) \to 0$ as $x \to -\infty$ (since $e^x \to 0$ as $x \to -\infty$).

Thus $y = 0$ is an asymptote for the left-hand part of the graph.

To locate the curve in the interval $(-\infty,0)$, we calculate $f(-1) \simeq -0.37$ and $f(-2) \simeq -0.07$.

Thus we can sketch the following graph.

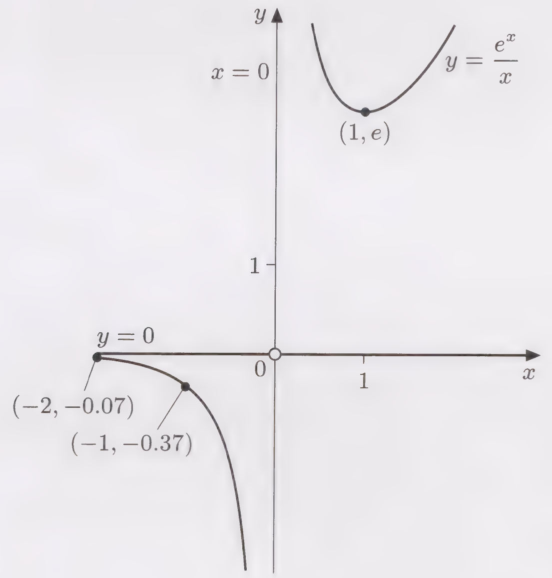


Figure S.6

(d) $f(x) = x^3 - 3x - 3$

Step 1: The domain of f is $\mathbb{R}$.

Step 2: The function f is neither even nor odd.

Step 3: The x-intercepts occur when $f(x) = 0$; that is, when

$$f(x) = x^3 - 3x - 3 = 0.$$

This cubic equation cannot be solved easily, but note that $f(2) = -1 < 0$ and $f(3) = 15 > 0$, so there is a zero of f between 2 and 3.

The y-intercept is $f(0) = -3$.

Step 4: The sign table for f is omitted, since we do not have accurate values for the zeros of f.

Step 5: The derivative of f is

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1),$$

which has the following sign table.

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$x-1$	$-$	$-$	$-$	0	$+$
$x+1$	$-$	0	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$	0	$+$

Thus f is increasing on $(-\infty,-1)$ and $(1,\infty)$, and decreasing on $(-1,1)$. Also, f has the following stationary points:

- ◇ $x = -1$, which is a local maximum with value $f(-1) = -1$;
- ◇ $x = 1$, which is a local minimum with value $f(1) = -5$.

Step 6: From Table 3.1, we know that

$$f(x) \to \infty \text{ as } x \to \infty$$

and

$$f(x) \to -\infty \text{ as } x \to -\infty.$$

Thus we can sketch the following graph.

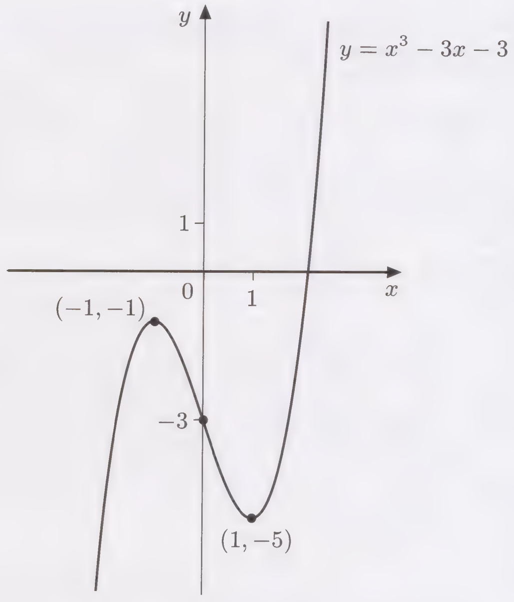


Figure S.7

(A good approximate value for the x-intercept is found in Exercise 4.1.)

Solution 4.1

(a) With

$$f(x) = x^3 - 2x - 2,$$

we have $f(1) = -3 < 0$ and $f(2) = 2 > 0$. Thus the function f changes sign in the interval $(1, 2)$, so it has a zero in that interval.

(b) Since $f'(x) = 3x^2 - 2$, the Newton–Raphson formula is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^3 - 2x_n - 2}{3x_n^2 - 2} \\ &= \frac{x_n\left(3x_n^2 - 2\right) - \left(x_n^3 - 2x_n - 2\right)}{3x_n^2 - 2} \\ &= \frac{3x_n^3 - 2x_n - x_n^3 + 2x_n + 2}{3x_n^2 - 2} \\ &= \frac{2x_n^3 + 2}{3x_n^2 - 2} \quad (n = 0, 1, 2, \ldots). \end{aligned}$$

(c) To maximum calculator accuracy:

$$\begin{aligned} x_0 &= 2, \\ x_1 &= 1.8, \\ x_2 &= 1.769\,948\,187, \\ x_3 &= 1.769\,292\,663, \\ x_4 &= 1.769\,292\,354. \end{aligned}$$

(d) We obtain with a calculator

$$f(x_4) = 0 \text{ (to 8 d.p.)}.$$

Thus x_4 is indeed a good approximation to a solution of $f(x) = 0$.

Solution 4.2

If $f(x) = x^2 - a$, then $f'(x) = 2x$, so the Newton–Raphson formula is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^2 - a}{2x_n} \\ &= \frac{2x_n^2 - \left(x_n^2 - a\right)}{2x_n} \\ &= \frac{x_n^2 + a}{2x_n} \\ &= \tfrac{1}{2}\left(x_n + \frac{a}{x_n}\right), \end{aligned}$$

as required.

Solution 4.3

(a) With $x_0 = 0.8165$ we have

$$x_1 = 184\,398.554\ldots.$$

This large value occurs because x_0 is close to the value $x = \sqrt{2/3} = 0.816\,496\,580\ldots$, for which $f'(x) = 3x^2 - 2 = 0$. In general, the Newton–Raphson method fails to converge to a zero of a function f if we choose x_0 to be a stationary point of f, or very close to such a point.

(For this value of x_0 the sequence x_n does tend to a zero of a function f, but it takes about 30 iterations to arrive close to the zero.)

(b) With $x_0 = 0$ we have $x_1 = -1$ and $x_2 = 0$. Thus the sequence x_n endlessly repeats the values $0, -1$; that is, it converges to the 2-cycle $0, -1$. In general, the Newton–Raphson method fails to converge to a zero of a function f if it converges to a p-cycle, for $p \geq 2$.

Solutions to Exercises

Solution 1.1

(a) We have, for $h \neq 0$,

$$\begin{aligned}
&\frac{f(x+h)-f(x)}{h}\\
&= \frac{((x+h)^3+2(x+h))-(x^3+2x)}{h}\\
&= \frac{x^3+3x^2h+3xh^2+h^3+2x+2h-x^3-2x}{h}\\
&= \frac{3x^2h+3xh^2+h^3+2h}{h}\\
&= 3x^2+3xh+h^2+2,
\end{aligned}$$

as required.

(b) By equation (1.2), we need to find the limit as h tends to 0 of the quotient from part (a):

$$\lim_{h\to 0}(3x^2+3xh+h^2+2) = 3x^2+2,$$

so

$$f'(x) = 3x^2+2.$$

Solution 1.2

(a) Here $g(t) = \ln t$, so

$$g'(t) = \frac{1}{t} \text{ and } g''(t) = -\frac{1}{t^2}.$$

(b) Here $s = t^3$, so

$$\frac{ds}{dt} = 3t^2,\ \frac{d^2s}{dt^2} = 6t,\ \frac{d^3s}{dt^3} = 6 \text{ and } \frac{d^4s}{dt^4} = 0.$$

Solution 2.1

(a) Here $f(x) = 2\ln x - 3\arcsin x$, so by the Sum and Constant Multiple Rules

$$f'(x) = \frac{2}{x} - \frac{3}{\sqrt{1-x^2}} \quad (0 < x < 1).$$

(b) Here $f(x) = x^3\tan x$, so by the Product Rule

$$f'(x) = 3x^2\tan x + x^3\sec^2 x.$$

(c) Here $g(t) = t^2/\cos(2t)$, so by the Quotient Rule

$$\begin{aligned}
g'(t) &= \frac{\cos(2t)(2t) - t^2(-2\sin(2t))}{\cos^2(2t)}\\
&= \frac{2t\cos(2t)+2t^2\sin(2t)}{\cos^2(2t)}\\
&= \frac{2t}{\cos^2(2t)}\left(\cos(2t)+t\sin(2t)\right).
\end{aligned}$$

(d) Here $f(x) = \sin(1/x)$, so by the Composite Rule

$$f'(x) = \cos(1/x)(-1/x^2) = -\frac{\cos(1/x)}{x^2}.$$

(e) Here $f(x) = \cos^2(3x)$, so by the Composite Rule

$$\begin{aligned}
f'(x) &= 2\cos(3x) \times \frac{d}{dx}(\cos(3x))\\
&= 2\cos(3x) \times (-3\sin(3x))\\
&= -6\cos(3x)\sin(3x).
\end{aligned}$$

(f) Here $g(y) = \dfrac{\ln(\cos y)}{y}$, so by the Quotient Rule and Composite Rule

$$\begin{aligned}
g'(y) &= \frac{y\dfrac{d}{dy}(\ln(\cos y)) - \ln(\cos y)\dfrac{d}{dy}(y)}{y^2}\\
&= \frac{y\left(\dfrac{1}{\cos y}\right)(-\sin y) - \ln(\cos y)}{y^2}\\
&= -\frac{y\tan y + \ln(\cos y)}{y^2}.
\end{aligned}$$

(g) Here $k(t) = \sin(t^2)\cos t$, so by the Product Rule and Composite Rule

$$\begin{aligned}
k'(t) &= \frac{d}{dt}(\sin(t^2))\cos t + \sin(t^2)\frac{d}{dt}(\cos t)\\
&= 2t\cos(t^2)\cos t - \sin t\sin(t^2).
\end{aligned}$$

(h) Here $f(x) = \arctan(e^{x^2})$, so by the Composite Rule, applied twice,

$$\begin{aligned}
f'(x) &= \frac{1}{1+(e^{x^2})^2}\frac{d}{dx}(e^{x^2})\\
&= \frac{1}{1+e^{2x^2}}(e^{x^2})(2x)\\
&= \frac{2xe^{x^2}}{1+e^{2x^2}}.
\end{aligned}$$

(i) The function

$$h(t) = 2\sin(t+1) - t\cos(t^2-1)$$

involves sums, constant multiples, a product and composites. Using the corresponding rules, we obtain

$$\begin{aligned}
h'(t) &= 2\cos(t+1)\\
&\quad - \left(1\times\cos(t^2-1) + t\left(-\sin(t^2-1)\times 2t\right)\right)\\
&= 2\cos(t+1) - \cos(t^2-1) + 2t^2\sin(t^2-1).
\end{aligned}$$

Solution 2.2

(a) By the Quotient Rule, we have

$$\begin{aligned}\frac{d}{dx}(\operatorname{cosec} x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right)\\ &= \frac{(\sin x)(0) - 1(\cos x)}{(\sin x)^2}\\ &= \frac{-\cos x}{\sin^2 x}\\ &= -\left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)\\ &= -\operatorname{cosec} x \cot x,\end{aligned}$$

as required.

(b) By the Composite Rule, we have

$$\begin{aligned}\frac{d}{dx}\left(\arcsin\left(\frac{x}{a}\right)\right) &= \frac{1}{\sqrt{1-(x/a)^2}} \times \frac{d}{dx}\left(\frac{x}{a}\right)\\ &= \frac{1}{a\sqrt{1-(x/a)^2}}\\ &= \frac{1}{\sqrt{a^2 - x^2}}, \quad \text{for } -a < x < a,\end{aligned}$$

as required.

Solution 3.1

(a) $f(x) = \dfrac{2-x}{x+3}$.

Step 1: The denominator $x+3$ is 0 when $x=-3$, so the domain of f is $\mathbb{R}$ except for -3.

Step 2: The function f is neither even nor odd.

Step 3: The only x-intercept occurs when $2-x=0$; that is, $x=2$. The y-intercept is $f(0)=\frac{2}{3}$.

Step 4: Here is a sign table for $f(x)$.

x	$(-\infty,-3)$	-3	$(-3,2)$	2	$(2,\infty)$
$2-x$	$+$	$+$	$+$	0	$-$
$x+3$	$-$	0	$+$	$+$	$+$
$f(x)$	$-$	$*$	$+$	0	$-$

Thus f is positive on $(-3,2)$, and negative on $(-\infty,-3)$ and $(2,\infty)$.

Step 5: The derivative of f is

$$f'(x) = \frac{(x+3)(-1)-(2-x)1}{(x+3)^2} = \frac{-5}{(x+3)^2},$$

so $f'(x)<0$ for all x in the domain of f. Thus f is decreasing on $(-\infty,-3)$ and $(-3,\infty)$.

Step 6: Since the denominator is 0 when x is -3, the line $x=-3$ is a vertical asymptote. Also

$$\begin{aligned}f(x) = \frac{2-x}{x+3} &= \frac{2/x-1}{1+3/x}\\ &\to \frac{0-1}{1+0} = -1 \text{ as } x \to \pm\infty.\end{aligned}$$

So the line $y=-1$ is a horizontal asymptote.

To locate the curve in the interval $(-\infty,-3)$, we calculate $f(-4)=-6$ and $f(-5)=-\frac{7}{2}=-3\frac{1}{2}$.

Thus we can sketch the following graph.

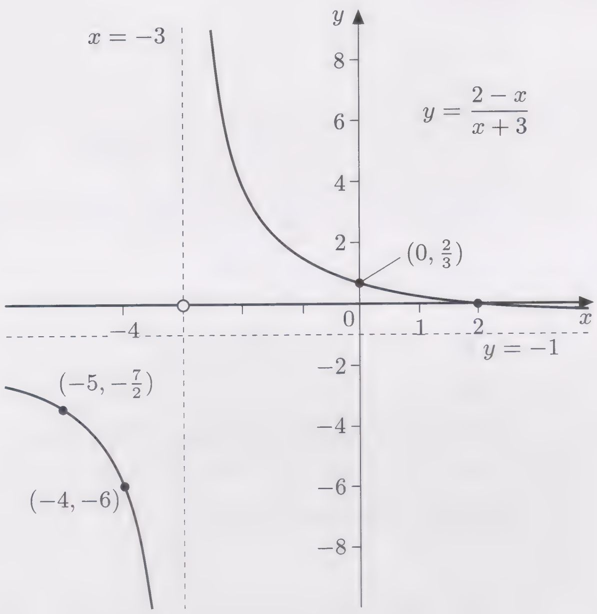


Figure S.8

(b) $f(x) = \dfrac{x^2}{x^2+5}$

Step 1: The domain of f is $\mathbb{R}$, since $x^2+5 \neq 0$.

Step 2: The function f is even because

$$f(-x) = \frac{(-x)^2}{(-x)^2+5} = \frac{x^2}{x^2+5} = f(x).$$

Step 3: The only x-intercept occurs when $x^2=0$; that is, $x=0$. The y-intercept is $f(0)=0$.

Step 4: Without constructing a sign table, we can say that f is positive on $(0,\infty)$ and $(-\infty,0)$.

Step 5: The derivative of f is

$$\begin{aligned}f'(x) &= \frac{(x^2+5)(2x)-(x^2)(2x)}{(x^2+5)^2}\\ &= \frac{10x}{(x^2+5)^2}.\end{aligned}$$

Without constructing a sign table, we can say that f' is positive on $(0,\infty)$ and negative on $(-\infty,0)$. Also, f has a stationary point at 0.

Thus f is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$. The stationary point at 0 is a local minimum with value $f(0)=0$.

Step 6: Since the denominator is never 0 there are no vertical asymptotes. Also

$$\begin{aligned}f(x) = \frac{x^2}{x^2+5} &= \frac{1}{1+5/x^2}\\ &\to \frac{1}{1+0} = 1 \text{ as } x \to \pm\infty.\end{aligned}$$

So the line $y = 1$ is a horizontal asymptote.

Thus we can sketch the following graph, in which unequal scales have been employed again.

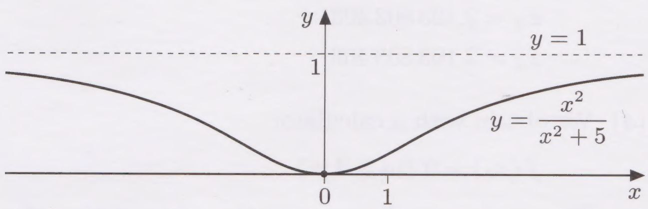


Figure S.9

(c) $f(x) = \dfrac{e^{3x}}{(x+1)^2}$

Step 1: The denominator $(x+1)^2$ is 0 when $x = -1$, so the domain of f is $\mathbb{R}$ except for -1.

Step 2: The function f is neither even nor odd.

Step 3: Since $e^{3x} \neq 0$, for $x \in \mathbb{R}$, there are no x-intercepts. The y-intercept is $f(0) = 1$.

Step 4: Here is a sign table for $f(x)$.

x	$(-\infty,-1)$	-1	$(-1,\infty)$
e^{3x}	$+$	$+$	$+$
$(x+1)^2$	$+$	0	$+$
$f(x)$	$+$	$*$	$+$

Thus f is positive on $(-1,\infty)$ and $(-\infty,-1)$.

Step 5: The derivative of f is

$$\begin{aligned} f'(x) &= \frac{(x+1)^2 3e^{3x} - e^{3x} \times 2(x+1)}{(x+1)^4} \\ &= \frac{e^{3x}(x+1)(3(x+1)-2)}{(x+1)^4} \\ &= \frac{e^{3x}(3x+1)}{(x+1)^3}, \end{aligned}$$

which has the following sign table.

x	$(-\infty,-1)$	-1	$(-1,-\frac{1}{3})$	$-\frac{1}{3}$	$(-\frac{1}{3},\infty)$
e^{3x}	$+$	$+$	$+$	$+$	$+$
$3x+1$	$-$	$-$	$-$	0	$+$
$(x+1)^3$	$-$	0	$+$	$+$	$+$
$f'(x)$	$+$	$*$	$-$	0	$+$

Thus f is increasing on $(-\infty,-1)$ and $(-\frac{1}{3},\infty)$, and decreasing on $(-1,-\frac{1}{3})$. Also, f has a stationary point at $-\frac{1}{3}$, which is a local minimum with value $f(-\frac{1}{3}) = \frac{9}{4}e^{-1} = 0.827\ldots$.

Step 6: The line $x = -1$ is a vertical asymptote. Also,

$f(x) \to \infty$ as $x \to \infty$ (given)

$f(x) \to 0$ as $x \to -\infty$ (since $e^{3x} \to 0$ as $x \to -\infty$).

To locate the curve in the interval $(-\infty,-1)$, we calculate $f(-1.1) \simeq 3.69$ and $f(-1.4) \simeq 0.09$.

Thus we can sketch the following graph.

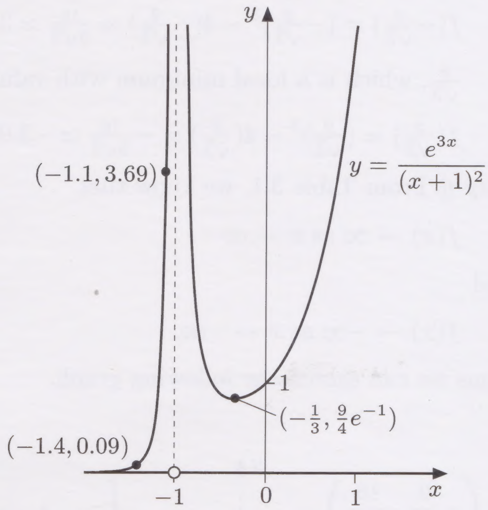


Figure S.10

(d) $f(x) = x^3 - 4x$

Step 1: The domain of f is $\mathbb{R}$.

Step 2: The function f is odd because

$$f(-x) = (-x)^3 - 4(-x) = -x^3 + 4x = -f(x).$$

Step 3: The x-intercepts occur when $f(x) = 0$; that is, when

$$f(x) = x^3 - 4x = x(x^2-4) = 0.$$

Thus the x-intercepts are $0, 2$ and -2.

The y-intercept is $f(0) = 0$.

Step 4: Here is a sign table for

$$f(x) = x^3 - 4x = x(x^2-4) = x(x-2)(x+2).$$

x	$(-\infty,-2)$	-2	$(-2,0)$	0	$(0,2)$	2	$(2,\infty)$
x	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x+2$	$-$	0	$+$	$+$	$+$	$+$	$+$
$f(x)$	$-$	0	$+$	0	$-$	0	$+$

Thus f is positive on $(-2,0)$ and $(2,\infty)$, and negative on $(-\infty,-2)$ and $(0,2)$.

Step 5: The derivative of f is

$$\begin{aligned} f'(x) &= 3x^2 - 4 \\ &= 3(x^2 - \tfrac{4}{3}) = 3\left(x - \tfrac{2}{\sqrt{3}}\right)\left(x + \tfrac{2}{\sqrt{3}}\right), \end{aligned}$$

which has the following sign table.

x	$(-\infty,-\frac{2}{\sqrt{3}})$	$-\frac{2}{\sqrt{3}}$	$(-\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}})$	$\frac{2}{\sqrt{3}}$	$(\frac{2}{\sqrt{3}},\infty)$
$x-\frac{2}{\sqrt{3}}$	$-$	$-$	$-$	0	$+$
$x+\frac{2}{\sqrt{3}}$	$-$	0	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$	0	$+$

Thus f is increasing on $(-\infty, -\frac{2}{\sqrt{3}})$ and $(\frac{2}{\sqrt{3}}, \infty)$, and decreasing on $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$. Also, f has the following stationary points:

◇ $-\frac{2}{\sqrt{3}}$, which is a local maximum with value

$$f(-\tfrac{2}{\sqrt{3}}) = (-\tfrac{2}{\sqrt{3}})^3 - 4(-\tfrac{2}{\sqrt{3}}) = \tfrac{16}{3\sqrt{3}} \simeq 3.08.$$

◇ $\frac{2}{\sqrt{3}}$, which is a local minimum with value

$$f(\tfrac{2}{\sqrt{3}}) = (\tfrac{2}{\sqrt{3}})^3 - 4(\tfrac{2}{\sqrt{3}}) = -\tfrac{16}{3\sqrt{3}} \simeq -3.08.$$

Step 6: From Table 3.1, we know that

$$f(x) \to \infty \text{ as } x \to \infty$$

and

$$f(x) \to -\infty \text{ as } x \to -\infty.$$

Thus we can sketch the following graph.

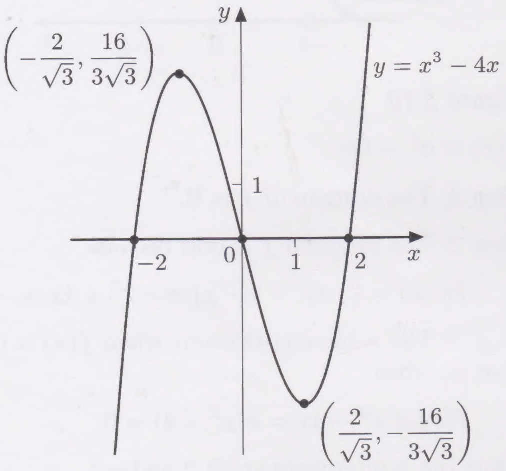


Figure S.11

Solution 4.1

(a) With

$$f(x) = x^3 - 3x - 3,$$

we have $f(2) = -1 < 0$ and $f(3) = 15 > 0$. Thus the function f changes sign in the interval $(2, 3)$, so it has a zero in that interval.

(b) Since $f'(x) = 3x^2 - 3$, the Newton–Raphson formula is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^3 - 3x_n - 3}{3x_n^2 - 3} \\ &= \frac{x_n\,(3x_n^2 - 3) - (x_n^3 - 3x_n - 3)}{3x_n^2 - 3} \\ &= \frac{2x_n^3 + 3}{3x_n^2 - 3} \quad (n = 0, 1, 2, \ldots). \end{aligned}$$

(c) To maximum calculator accuracy:

$$\begin{aligned} x_0 &= 2, \\ x_1 &= 2.111\,111\,111, \\ x_2 &= 2.103\,835\,979, \\ x_3 &= 2.103\,803\,403, \\ x_4 &= 2.103\,803\,403. \end{aligned}$$

(d) We obtain with a calculator

$$f(x_4) = 0 \text{ (to 8 d.p.)}.$$

Thus x_4 is a good approximation to a solution of $f(x) = 0$.

(e) The Newton–Raphson method fails in this case if we choose x_0 to be either of the stationary points of f, which are ± 1.

Solution 4.2

(a) With

$$f(x) = x - \cos x,$$

we have $f(0) = -1 < 0$ and $f(\frac{1}{2}\pi) = \frac{1}{2}\pi > 0$. Thus the function f changes sign in the interval $(0, \frac{1}{2}\pi)$, so it has a zero in that interval.

(b) Since $f'(x) = 1 + \sin x$, the Newton–Raphson formula is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n - \cos(x_n)}{1 + \sin(x_n)} \\ &= \frac{x_n(1 + \sin(x_n)) - (x_n - \cos(x_n))}{1 + \sin(x_n)} \\ &= \frac{x_n \sin(x_n) + \cos(x_n)}{1 + \sin(x_n)} \quad (n = 0, 1, 2, \ldots). \end{aligned}$$

(c) To maximum calculator accuracy:

$$\begin{aligned} x_0 &= 0, \\ x_1 &= 1, \\ x_2 &= 0.750\,363\,868, \\ x_3 &= 0.739\,112\,891, \\ x_4 &= 0.739\,085\,133. \end{aligned}$$

(d) We obtain with a calculator

$$f(x_4) = 0 \text{ (to 9 d.p.)}.$$

Thus x_4 is a good approximation to a solution of $f(x) = 0$.

(e) The Newton–Raphson method fails in this case if we choose x_0 to be a stationary point of f, that is, a solution of the equation

$$f'(x) = 1 + \sin x = 0,$$

such as $-\frac{1}{2}\pi$.

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